

C.1 · Single-variable calculus and applications

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Exact algebra, trigonometry in radians, functions and inequalities.

- Use limits, continuity and differentiability
- Apply derivatives, integrals and the fundamental theorem
- Recognise Riemann sums and distinguish them from infinite series

continuity : Agreement of a function value with its limit.

convergence : Approach to a finite limiting value.

Choose and justify a method

A limit describes values near a point, without requiring the function to be defined there. Continuity adds the requirement that the function value exists and agrees with the limit. Algebraic cancellation is valid only away from the cancelled zero, but can reveal a removable limit. One-sided limits must agree for a two-sided limit. For quotient limits, check the denominator and hypotheses before applying a rule; $0/0$ is an indeterminate form, not an answer.

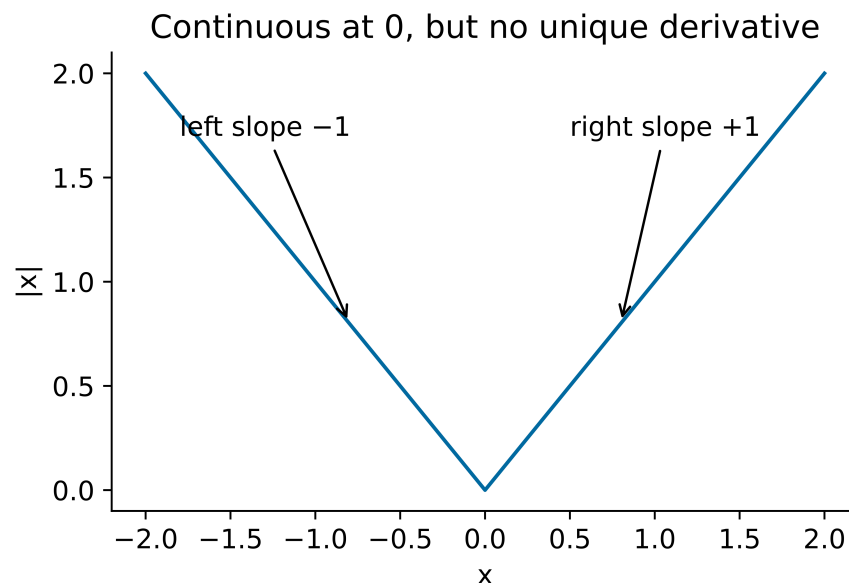
A derivative is a limit of difference quotients and implies continuity; the converse fails, as $|x|$ at zero shows. Product, quotient and chain rules describe different structures: the derivative of $f(g(x))$ is $f'(g(x))g'(x)$, not a product of unrelated values. Differentiating a composition a second time generally produces two terms. Check domains, nonzero denominators and differentiability assumptions before using a symbolic expression as a derivative.

The fundamental theorem says that the derivative of $\int$ from a to x of a continuous integrand $f(t)$ is $f(x)$. With a variable upper bound $h(x)$, multiply by $h'(x)$; with two variable bounds, subtract the corresponding lower-bound contribution. A definite integral also arises as a limit of Riemann sums. Rewrite the sum as $(1/n)\Sigma f(k/n)$ before identifying the integral on $[0,1]$, rather than treating n -dependent terms as constants.

For example Σ from $k=1$ to n of $n/(n^2+k^2)$ equals $(1/n)\Sigma 1/(1+(k/n)^2)$, tending to $\int_0^1 1/(1+t^2)dt=\pi/4$. This limit uses continuity and the partition width $1/n$. A series over an unbounded number of terms is a different limit: terms tending to zero do not alone guarantee convergence. For $\Sigma 1/k$ the partial sums diverge; compare, estimate or apply a valid series test instead of using the necessary term condition as sufficient.

Worked reasoning

For $F(x)=\int_0^{x^2} e^t dt$, the fundamental theorem and chain rule give $F'(x)=e^{x^2} \cdot 2x$. At $x=1$, $F'(1)=2e$. The upper limit is x^2 , so omitting $2x$ misses its rate of change.



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

A-level calculus is useful prerequisite material but does not cover the undergraduate analysis and applications tested here.

Guided application

Let $F(x) = \int_x^{x^2} (1+t^2) dt$. Find $F'(x)$ and $F'(2)$. State why the theorem applies.

■ Worked solution

The integrand is continuous on the real line. The fundamental theorem and chain rule apply at both bounds:

$$F'(x) = (1+x^4)2x - (1+x^2).$$

$$F'(2) = (1+2^4)2 \cdot 2 - (1+2^2) = 63.$$

Independently, an antiderivative is $t + t^3/3$. Thus $F = x^2 + x^6/3 - x - x^3/3$, with the same derivative.

Independent transfer

Find $\lim_{n \rightarrow \infty} \sum_{k=1}^n k/(n^2 + k^2)$. Explain why this is a Riemann-sum limit. Then decide whether $\sum_{k=1}^{\infty} 1/k$ converges.

■ Check after attempting

Rewrite each term as $\frac{1}{n} \frac{k/n}{1+(k/n)^2}$. The continuous function $f(t) = t/(1+t^2)$ on $[0, 1]$ and mesh $1/n$ give

$$L = \int_0^1 \frac{t}{1+t^2} dt = \frac{1}{2} \ln 2.$$

The harmonic series diverges: the block from $2^{j-1} + 1$ through 2^j contains 2^{j-1} terms at least $1/2^j$, so each block adds at least $1/2$. Terms tending to zero is necessary but insufficient. The finite-sum integrand here depends on the scaled index; it is not the harmonic series.