

# 9.1 Groups, cosets and quotient maps

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_ **Total: 10 marks**

**KEY WORDS** coset • normal subgroup

## Objective

Verify group structure and compute element orders and subgroup indices; Use kernels and images to identify quotient groups; Distinguish normal subgroups from arbitrary subgroups; Classify permutation conjugacy by cycle type.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

## 1 Key knowledge and worked examples

**coset** : A translate of a subgroup that forms one part of the coset partition.

**normal subgroup** : A subgroup invariant under conjugation by every group element.

Before computing an order, check closure, associativity, an identity and an inverse for every element under the stated operation. A subset can inherit associativity yet fail closure or omit the identity. In a finite group, the order of an element is the least positive power giving the identity. In the additive group  $\mathbb{Z}/n\mathbb{Z}$ , it is the least positive multiple giving zero; the order of residue  $a$  is  $n/\gcd(a,n)$ . Lagrange’s theorem says subgroup orders divide the group order. The converse is not a general existence theorem, and the order of a group is not the order of each element.

Define  $\phi$  from  $\mathbb{Z}/12\mathbb{Z}$  to  $\mathbb{Z}/3\mathbb{Z}$  by reducing residues modulo 3. It is onto and preserves addition. Its kernel  $H$  is  $\{0,3,6,9\}$ , so  $|H|=4$  and the index is  $12/4=3$ . The other cosets are  $\{1,4,7,10\}$  and  $\{2,5,8,11\}$ . Thus  $(\mathbb{Z}/12\mathbb{Z})/H$  is isomorphic to  $\mathbb{Z}/3\mathbb{Z}$ . The element 3 in the original group has order  $12/\gcd(3,12)=4$ , not 3.

Reduction modulo 3: three fibres, four elements each

| Quotient class | Elements of $\mathbb{Z}/12\mathbb{Z}$ | Image in $\mathbb{Z}/3\mathbb{Z}$ |
|----------------|---------------------------------------|-----------------------------------|
| $H$            | 0, 3, 6, 9                            | 0                                 |
| $1 + H$        | 1, 4, 7, 10                           | 1                                 |
| $2 + H$        | 2, 5, 8, 11                           | 2                                 |

*Original course illustration; its values belong to the worked example, not the later practice.*

## 2 Practice

2.1 What is the order of 8 in the additive group  $\mathbb{Z}/20\mathbb{Z}$ ? [1]

- A 2
- B 4
- C 5
- D 8
- E 20

2.2 A homomorphism from a finite group of order 30 has kernel of order 5. How many elements are in its image? [1]

- A 5
- B 6
- C 10
- D 25
- E 30

**2.3** In the additive group  $\mathbb{Z}/18\mathbb{Z}$ , let  $H = \langle 6 \rangle$ . List H and its cosets. Determine the order of 6 and identify the quotient. [4]

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**3 Exam-style questions**

**3.1** In  $S_3$ , let  $H = \{e, (12)\}$ . Check whether  $H$  is normal, and decide whether coset multiplication defines a quotient group. Separately find the order of  $(123)(45)$  in  $S_5$ . [4]

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