

Solutions

2.1 B. It is both closed and bounded in the real line. A omits its limit endpoints; C is unbounded. Both fail a Euclidean compactness condition.

2.2 A. Convergence gives an eventual bound plus finitely many earlier terms. B fails for a constant or oscillating convergent sequence; C fails for the constant sequence one-half.

2.3 Given $\epsilon > 0$, choose $\delta = \min(1, \epsilon/5)$. If $|x - 2| < \delta$, then $1 < x < 3$, so $|x + 2| < 5$.

$$|x^2 - 4| = |x - 2||x + 2| < 5\delta \leq \epsilon.$$

The choice depends on epsilon, not on the later input x . The interval $[1, 3]$ is nonempty and compact in the ordinary real metric. Continuity gives attained extrema. Here $x^2 \geq 1$, with equality at 1, so the minimum is 1.

3.1 X is bounded and closed relative to itself, since its complement in X is empty. The Cauchy sequence $1/(n + 2)$ has no limit in X , so X is not complete and therefore not compact as a metric space. Directly, the relative open cover $\{(1/n, 1) : n \geq 2\}$ covers X but has no finite subcover. Heine–Borel requires closed and bounded as a subset of Euclidean space, not merely closed in an arbitrary chosen space. X is not closed in $\mathbb{R}$. No theorem hypothesis may change its ambient space halfway through the argument.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.