

Solutions

2.1 A. Addition has identity 0 and inverse $-a$. B fails because integer division is not closed, for example $1/2$; C fails because zero is the additive identity.

2.2 C. $\gcd(2,9)=1$; its inverse is 5. Three and six share a factor three with nine; neither product can be one modulo nine. Two times five has remainder one.

2.3 Both whole residue sets are additive groups: closure, associativity and identity come from integer addition, and $-a$ is an additive inverse. A multiplicative residue is a unit precisely when its greatest common divisor with the modulus is one. The units modulo five are 1, 2, 3, 4; modulo six they are 1, 5. Modulo six, $2 \cdot 3 = 0$ although both factors are nonzero. The nonzero set is not even closed under multiplication. Modulo five every nonzero residue has an inverse; this ring is a field. A ring, its additive group and its group of units are different structures.

3.1 The map preserves addition and multiplication. Its kernel is $6\mathbb{Z}$ and its image is all six residues. The quotient identifies integers with the same remainder, so $\mathbb{Z}/\ker\phi \cong \mathbb{Z}/6\mathbb{Z}$. The congruence means $6 \mid 2(x-1)$, equivalently $3 \mid x-1$. Hence $x \equiv 1 \pmod{3}$, giving residues 1 and 4 modulo six. Both substitute correctly. Cancelling 2 while retaining modulus six would discard 4; 2 has no multiplicative inverse there.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.