

3.1 Linear algebra

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS rank • eigenvalue

Objective

Relate rank, nullity and solutions of linear systems; Analyse vector spaces and linear transformations; Compute eigenvalues, determinants and diagonalisation conditions.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

rank : Dimension of the image of a linear map.

eigenvalue : A scalar satisfying $A\mathbf{v}=\lambda\mathbf{v}$ for a nonzero $\mathbf{v}$.

A real vector space is closed under its specified addition and scalar multiplication and satisfies the vector-space axioms. A basis is an independent spanning list: every vector has a unique coordinate representation in that list. To test independence of matrix columns, solve the homogeneous system $A\mathbf{c}=\mathbf{0}$; only the zero coefficient vector means independence. A spanning set can contain redundant vectors, so spanning alone does not make a basis.

$A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ has characteristic polynomial $(1-\lambda)^2$. Eigenvectors satisfy $y=0$, so its eigenspace has dimension one. Two independent eigenvectors are needed to diagonalise a 2×2 matrix; A is not diagonalizable.

A repeated eigenvalue can have different eigenspace dimensions

Matrix	Eigenvalue	Dimension	Diagonalizable?
$\begin{bmatrix}1 & 1 \\ 0 & 1\end{bmatrix}$	1, twice	1	No
$\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$	1, twice	2	Yes

Original course illustration; its values belong to the worked example, not the later practice.

2 Practice

2.1 A 3-column matrix has rank 2. Its nullity is:

[1]

- A 1
- B 2
- C 3
- D 0
- E 5

2.2 $\det A=0$ implies:

[1]

- A A is invertible
- B A is singular
- C All entries are zero
- D Every column is zero
- E Every eigenvalue is nonzero

2.3 For $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with $T(x, y, z) = (x + 2y + z, 2x + 4y + 2z)$, find bases of the kernel and image. Solve $T(v) = (3, 6)$ and $T(v) = (3, 7)$. [4]

3 Exam-style questions

3.1 Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Determine its eigenvalues, determinant and diagonalisation over $\mathbb{R}$ and $\mathbb{C}$. Find A^n for integers $n \geq 1$ without repeated multiplication. [4]
