

Solutions

2.1 A. Rank-nullity gives $3-2=1$. B, C and E exceed the one free column allowed by rank two; D would require three independent columns.

2.2 B. A zero determinant means the square matrix is singular. It can have many nonzero entries; a single dependence is enough. A and E contradict singularity. C and D are stronger than dependence and fail, for example, for $\text{diag}(1,0)$.

2.3 The second row is twice the first, so rank is one. The homogeneous equation gives $x = -2y - z$.

$$\ker T = \text{span}\{(-2, 1, 0), (-1, 0, 1)\}.$$

The two displayed vectors are independent. Every output is $(s, 2s)$, and $(1, 2)$ occurs; this is an image basis. Rank plus nullity is $1 + 2 = 3$, the domain dimension. For $(3, 6)$, all solutions are $(3 - 2a - b, a, b)$, with $a, b \in \mathbb{R}$. For $(3, 7)$ there is no solution, since $7 \neq 2 \cdot 3$.

3.1 The characteristic polynomial is $(2 - \lambda)^2$ and $\det A = 4$. Solving $(A - 2I)v = 0$ gives $v_2 = 0$, so the eigenspace has dimension one over either field. Two independent eigenvectors are required; the matrix is invertible but not diagonalizable over either field. Write $A = 2I + N$ where $N^2 = 0$. The commuting binomial expansion leaves only two terms:

$$A^n = 2^n I + n2^{n-1}N = \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}.$$

Multiplying this expression by A gives the formula at $n + 1$, independently checking the coefficient.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.