

27.1 Implicit differentiation and the inverse Jacobian

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS Jacobian matrix • local inverse

Objective

Differentiate a coupled implicit system as a linear system; Use a nonzero Jacobian determinant to justify a local inverse; Distinguish inverse partial derivatives from scalar reciprocal rules.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

Jacobian matrix : The matrix of first partial derivatives of a vector-valued map.

local inverse : An inverse defined on neighbourhoods of a particular input and output point.

For a differentiable map $F(u,v)=(x,y)=(f(u,v),g(u,v))$, its Jacobian is $J=[f_u, f_v, g_u, g_v]$. Small changes satisfy $[dx, dy] \text{ (transpose)} = J[du, dv] \text{ (transpose)}$ to first order. To find u_x while holding y fixed, differentiate both defining equations with respect to x : $f_u u_x + f_v v_x = 1$ and $g_u u_x + g_v v_x = 0$. These are a coupled linear system, not two independent scalar inverse rules.

Take $x=u^2+v$ and $y=u-v$. At $(u,v)=(1,0)$, the output is $(1,1)$, and $J=[[2,1],[1,-1]]$ has determinant -3 . The inverse is $[[1/3, 1/3], [1/3, -2/3]]$, so $u_x=1/3$ and $v_y=-2/3$ there. Multiplying J by this inverse gives the identity. The reciprocal $1/f_u=1/2$ is not u_x because changing u also requires changing v to keep y fixed.

Coupled sensitivity: inverse entries differ from reciprocals

Forward map	Jacobian	Inverse Jacobian
$x = u + v; y = u + 2v$	$[[1, 1], [1, 2]]$	$[[2, -1], [-1, 1]]$
$x = u^2 + v; y = u - v$ at $(u, v) = (1, 0)$	$[[2, 1], [1, -1]]$	$[[1/3, 1/3], [1/3, -2/3]]$

Original course illustration; its values belong to the worked example, not the later practice.

2 Practice

2.1 For $x=u+v, y=u+2v$, what is u_x when y is held fixed?

[1]

- A 0
- B $1/2$
- C 1
- D 2
- E -1

2.2 $J=[[2,1],[1,-1]]$. What is v_y in the inverse derivative matrix?

[1]

- A $-2/3$
- B $-1/2$
- C $1/3$
- D $1/2$
- E 2

2.3 Let $x = u^2 + v, y = u - v$. At $(u, v) = (1, 0)$, find the inverse Jacobian and the four inverse partial derivatives. Check by multiplication. [4]

3 Exam-style questions

3.1 For $F(u, v) = (u^2 - v^2, 2uv)$, decide whether its derivative is invertible away from the origin and whether F is globally injective on the punctured plane. What does a zero determinant at the origin establish by itself? [4]
