

Solutions

2.1 C. Only 1,3,7,9 are coprime to 10; each has a multiplicative inverse. There are four units. Two or three omit units; five or nine include a residue divisible by two or five.

2.2 D. $\mathbb{Z}/7\mathbb{Z}$ is a field because 7 is prime. The composite quotients have zero divisors, and $\mathbb{Z}/0\mathbb{Z}$ is $\mathbb{Z}$, not a field. The zero ideal has quotient $\mathbb{Z}$, which is not a field. Quotients by four, six and nine have nonzero zero divisors, so those ideals are not maximal.

2.3 Units are the residues coprime to 12: 1, 5, 7, 11. The remaining nonzero residues 2, 3, 4, 6, 8, 9, 10 are zero divisors. For each such a , $a \cdot (12/\gcd(a, 12)) = 0$ modulo 12, with the second factor nonzero. The ideal $(4) = \{0, 4, 8\}$ gives quotient isomorphic to $\mathbb{Z}/4\mathbb{Z}$. This is not a domain, since $2 \cdot 2 = 0$, and is not a field. Hence the ideal is neither prime nor maximal.

3.1 A nonzero free $\mathbb{Z}$ -module has no nonzero vector annihilated by a nonzero integer: examine each integer coordinate. But $6[1] = 0$ in $\mathbb{Z}/6\mathbb{Z}$. It is not free; a field-style basis argument is invalid. $\mathbb{Z}/(2) \cong \mathbb{F}_2$ is a field, so (2) is maximal and prime. Evaluation at zero maps $\mathbb{Q}[x]$ onto $\mathbb{Q}$ with kernel (x) ; its quotient is a field too. Both arguments use proper ideals in commutative rings with identity.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.