

9.1 Groups, cosets and quotient maps

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS coset • normal subgroup

Objective

Verify group structure and compute element orders and subgroup indices; Use kernels and images to identify quotient groups; Distinguish normal subgroups from arbitrary subgroups; Classify permutation conjugacy by cycle type.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

coset : A translate of a subgroup that forms one part of the coset partition.

normal subgroup : A subgroup invariant under conjugation by every group element.

Before computing an order, check closure, associativity, an identity and an inverse for every element under the stated operation. A subset can inherit associativity yet fail closure or omit the identity. In a finite group, the order of an element is the least positive power giving the identity. In the additive group $\mathbb{Z}/n\mathbb{Z}$, it is the least positive multiple giving zero; the order of residue a is $n/\gcd(a,n)$. Lagrange’s theorem says subgroup orders divide the group order. The converse is not a general existence theorem, and the order of a group is not the order of each element.

Define ϕ from $\mathbb{Z}/12\mathbb{Z}$ to $\mathbb{Z}/3\mathbb{Z}$ by reducing residues modulo 3. It is onto and preserves addition. Its kernel H is $\{0,3,6,9\}$, so $|H|=4$ and the index is $12/4=3$. The other cosets are $\{1,4,7,10\}$ and $\{2,5,8,11\}$. Thus $(\mathbb{Z}/12\mathbb{Z})/H$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The element 3 in the original group has order $12/\gcd(3,12)=4$, not 3.

Reduction modulo 3: three fibres, four elements each

Quotient class	Elements of $\mathbb{Z}/12\mathbb{Z}$	Image in $\mathbb{Z}/3\mathbb{Z}$
H	0, 3, 6, 9	0
$1 + H$	1, 4, 7, 10	1
$2 + H$	2, 5, 8, 11	2

Original course illustration; its values belong to the worked example, not the later practice.

2 Practice

2.1 What is the order of 8 in the additive group $\mathbb{Z}/20\mathbb{Z}$? [1]

- A 2
- B 4
- C 5
- D 8
- E 20

2.2 A homomorphism from a finite group of order 30 has kernel of order 5. How many elements are in its image? [1]

- A 5
- B 6
- C 10
- D 25
- E 30

2.3 In the additive group $\mathbb{Z}/18\mathbb{Z}$, let $H = \langle 6 \rangle$. List H and its cosets. Determine the order of 6 and identify the quotient. [4]

3 Exam-style questions

3.1 In S_3 , let $H = \{e, (12)\}$. Check whether H is normal, and decide whether coset multiplication defines a quotient group. Separately find the order of $(123)(45)$ in S_5 . [4]

Solutions

2.1 C. The least positive k with $8k$ divisible by 20 is 5; equivalently $20/\gcd(8,20)=5$. Two, four and eight multiples of eight are not zero modulo twenty; twenty works but is not the least positive count.

2.2 B. Fibres are kernel cosets of size 5, so the image has $30/5=6$ elements regardless of the codomain size. Five is the kernel size; ten, twenty-five and thirty do not equal the number of five-element fibres in thirty elements.

2.3 $H = \{0, 6, 12\}$ has size three. The six cosets are $r + H = \{r, r + 6, r + 12\}$ for $r = 0, 1, 2, 3, 4, 5$. They partition all 18 residues. The element 6 has order three, while the quotient has order six. Reduction modulo six is onto with kernel H , so $(\mathbb{Z}/18\mathbb{Z})/H \cong \mathbb{Z}/6\mathbb{Z}$. Normality holds because the original group is abelian.

3.1 Conjugation by (123) sends (12) to (23) , which is outside H . Thus H is not normal. Left cosets still partition S_3 , but multiplication of those cosets is not well defined independently of representatives. The disjoint cycles in $(123)(45)$ have lengths three and two, so its order is $\text{lcm}(3, 2) = 6$. Element order and subgroup index are different quantities.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.