

7.1 Complex analysis and residues

Name: _____ Class: _____ Date: _____ Total: 10 marks

KEY WORDS analytic • residue

Objective

Test complex differentiability; Use contour integrals and residues; Evaluate contour integrals with simple and higher-order pole residues.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

analytic : Complex differentiable throughout a neighbourhood.

residue : Coefficient of $(z-a)^{-1}$ in a Laurent series.

Write $f(z)=u(x,y)+iv(x,y)$. Complex differentiability imposes $u_x=v_y$ and $u_y=-v_x$; with continuous first partials locally, the Cauchy–Riemann equations establish analyticity there. They differ from real differentiability of a two-coordinate map. The conjugate function $x-iy$ fails these equations on every open neighbourhood, although real partial derivatives exist. State the region being checked, not only a convenient point.

For $f(z)=\text{conjugate}(z)$, $u=x$ and $v=-y$. Then $u_x=1$ but $v_y=-1$, so f is not complex differentiable. For $1/(z-2)$, a positively oriented circle $|z-2|=1$ encloses a simple pole of residue 1, hence the integral is $2\pi i$.

The residue is the coefficient of $(z - 1)^{-1}$

Function	Pole order	Residue at 1
$3/(z - 1)$	1	3
$e^z/(z - 1)^2$	2	e
$e^{2z}/(z - 1)^3$	3	$2e^2$

Original course illustration; its values belong to the worked example, not the later practice.

2 Practice

2.1 The residue of $3/(z-1)$ at $z=1$ is: [1]

- A 1
- B 3
- C 0
- D $2\pi i$
- E $6\pi i$

2.2 A positively oriented circle enclosing only that pole gives integral: [1]

- A 0
- B $3\pi i$
- C $6\pi i$
- D 3
- E $-6\pi i$

2.3 Use the Cauchy–Riemann equations to compare z^2 and $\bar{z}$. Find the residue of e^z/z^2 at zero and its integral around the counterclockwise unit circle. [4]

3 Exam-style questions

3.1 Integrate $1/[z(z-2)]$ around $|z|=1$, first counterclockwise and then clockwise. Can the same residue-theorem argument be used on $|z|=2$? [4]

Solutions

2.1 B. The coefficient of the simple-pole term is 3. One is the pole location, not its coefficient. Zero loses the pole term; $2\pi i$ and $6\pi i$ are integral-like quantities, not the Laurent coefficient three.

2.2 C. Multiply residue 3 by $2\pi i$. Zero omits the pole; $3\pi i$ misses the factor two; three is the residue; negative $6\pi i$ reverses orientation.

2.3 For z^2 , $u = x^2 - y^2$ and $v = 2xy$: $u_x = 2x = v_y$ and $u_y = -2y = -v_x$. The partials are continuous everywhere, so the function is analytic everywhere. For $\bar{z}$, $u = x$, $v = -y$ and $u_x = 1 \neq -1 = v_y$; it is nowhere complex differentiable. Since $e^z = 1 + z + z^2/2 + \cdots$, the coefficient of z^{-1} in e^z/z^2 is 1. Therefore the residue is 1 and the integral is $2\pi i$. The double pole does not force a zero residue.

3.1 Only zero is inside the unit circle. Its residue is $\lim_{z \rightarrow 0} 1/(z - 2) = -1/2$. Thus the counterclockwise integral is $-\pi i$ and the clockwise integral is πi . The pole at 2 is outside and contributes nothing. The circle $|z| = 2$ passes through a pole. The ordinary contour integral is not defined by this formula; the theorem requires analyticity on the contour. An indentation or principal-value prescription would be a different, explicitly specified problem.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.