

28.1 LU factorisation and nullity of composed maps

Name: _____ Class: _____ Date: _____ Total: 10 marks

KEY WORDS forward substitution • nullity

Objective

Solve a factored linear system by forward and backward substitution; Apply row permutations consistently when pivoting is required; Bound the kernel dimension of a composition using image-kernel intersections.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

forward substitution : Solving a lower triangular system from its first equation downward.

nullity : The dimension of the kernel of a linear map.

A factorisation $A=LU$ expresses a square matrix as lower triangular L and upper triangular U . With unit diagonal L , forward substitution is especially simple. An invertible matrix need not admit this form without row exchanges: a zero leading pivot can require a permutation. For $PA=LU$, solve $Ly=Pb$, then $Ux=y$. The permutation acts on the right side as well as the coefficient matrix. Nonzero leading principal pivots justify the usual no-exchange elimination, not invertibility alone.

Let $L=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$, $U=\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, and $b=(-1,-1,12)$. Forward substitution gives $y=(-1,1,8)$. Back substitution gives $x_3=2$, $x_2=(1-4)/3=-1$, and $x_1=(-1-(-1)+2)/2=1$. Therefore $x=(1,-1,2)$. Multiplying Ux gives y and multiplying Ly gives b , independently checking the factor order.

LU solve: forward through L , then backward through U

Stage	Known right side	Result
$Ly = b$	$b = (-1, -1, 12)$	$y = (-1, 1, 8)$
$Ux = y$	$y = (-1, 1, 8)$	$x = (1, -1, 2)$

Original course illustration; its values belong to the worked example, not the later practice.

2 Practice

2.1 For the stated unit-diagonal L and $b=(-1,-1,12)$, what is the second coordinate of y solving $Ly=b$? [1]

- **A** -3
- **B** -1
- **C** 0
- **D** 1
- **E** 12

2.2 A,B are maps $R^6 \rightarrow R^6$ with nullity $A=2$ and nullity $B=3$. What range is possible for nullity of $A \circ B$? [1]

- **A** 0 to 2
- **B** 2 to 3
- **C** 3 to 5
- **D** 5 to 6
- **E** Always 5

2.3 Let $L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $b = (5, 12)^T$. Solve $LUx=b$ and verify in the multiplied matrix. For $PA=LU$, where does P act when solving $Ax=b$? [4]

3 Exam-style questions

3.1 Let A and B be endomorphisms of $\mathbb{R}^5$ with nullities two and three, respectively. Find all possible nullities of AB, and justify both the bound and attainability. [4]

Solutions

2.1 D. $y_1 = -1$ and the second row is $2y_1 + y_2 = -1$. Therefore $y_2 = 1$. Substituting $y_1 = -1$ in $2y_1 + y_2 = -1$ gives $y_2 = 1$. Minus three, minus one, zero and twelve fail that second row.

2.2 C. Nullity B contributes 3, and $\text{im } B \cap \ker A$ has dimension from 0 to 2 in this ambient space. Nullity cannot be below three because $\ker B$ is included. It cannot exceed five because at most two new kernel dimensions arise from $\ker A$. Always five ignores images disjoint from $\ker A$.

2.3 First solve $Ly = b$: $y_1 = 5$, $2y_1 + y_2 = 12$, so $y_2 = 2$. Then $Ux = y$ gives $x_2 = 1$ and $3x_1 + 1 = 5$, hence $x_1 = 4/3$. The product matrix is $A = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$; applying it to $(4/3, 1)^T$ gives $(5, 12)^T$. If $PA = LU$, multiply the original equation by P and solve $Ly = Pb$, followed by $Ux = y$. Omitting P on b solves a different system.

3.1 Rank B is two. Restrict B to $\ker(AB)$ to get

$$\dim \ker(AB) = \dim \ker B + \dim(\text{im } B \cap \ker A).$$

The two subspaces in the intersection both have dimension two in five dimensions, so intersection dimension may be zero, one or two. Thus nullity is 3, 4 or 5. To realise each, take $\ker A = \text{span}(e_1, e_2)$ and image B respectively $\text{span}(e_3, e_4)$, $\text{span}(e_1, e_3)$ or $\text{span}(e_1, e_2)$. Choose any rank-two B onto that image. These constructions prove attainability rather than only a loose interval bound.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.