

13.1 Sequences, series and uniform convergence

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS Cauchy sequence • uniform convergence

Objective

Use Cauchy, monotone convergence and subsequence criteria; Distinguish pointwise from uniform convergence of functions; Check hypotheses before interchanging limits and integrals.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

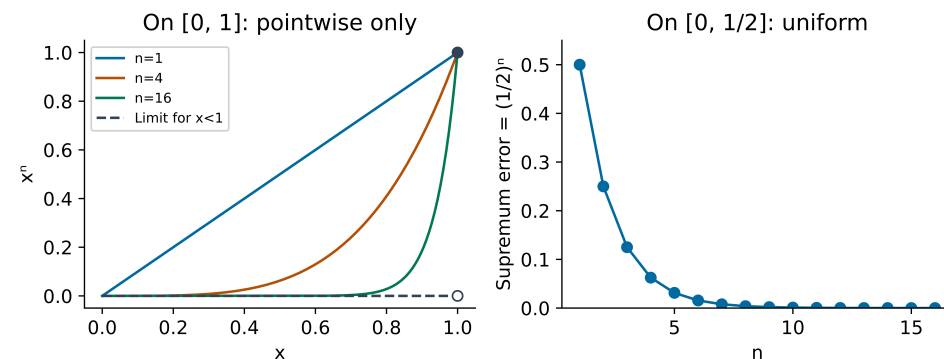
1 Key knowledge and worked examples

Cauchy sequence : A sequence whose sufficiently late terms are arbitrarily close to one another.

uniform convergence : Convergence with one error threshold index valid at every point of the domain.

A convergent real sequence is Cauchy, and every real Cauchy sequence converges because $\mathbb{R}$ is complete. In $\mathbb{Q}$ a Cauchy sequence can approach an irrational number and fail to converge within $\mathbb{Q}$. A bounded monotone real sequence converges. A bounded sequence need not converge, but Bolzano–Weierstrass guarantees a convergent subsequence; $(-1)^n$ has two different subsequential limits.

On $[0,1]$, $f_n(x)=x^n$ tends to 0 for $x<1$ and to 1 at $x=1$. The limit is discontinuous, so convergence cannot be uniform; directly, the supremum error is 1, approached below 1. On $[0,a]$ with $0\leq a<1$, the limit is zero and the supremum is a^n , which tends to zero, so convergence is uniform. Changing the domain changes the conclusion.



Original course illustration; its values belong to the worked example, not the later practice.

2 Practice

2.1 Which statement must hold for a bounded real sequence? [1]

- A It converges
- B It is monotone
- C It has a convergent subsequence
- D It is Cauchy
- E Its terms tend to zero

2.2 For $f_n(x)=\sin(nx)/n$ on $\mathbb{R}$, convergence to zero is what type? [1]

- A Not pointwise
- B Pointwise but not uniform
- C Uniform
- D Only at integer x
- E Divergent at zero

2.3 Determine pointwise and uniform convergence of $f_n(x) = x^n$ on $[0, 1]$ and on $[0, a]$ for fixed $0 < a < 1$. Compare $\sum 1/n^2$ with the alternating harmonic series for absolute convergence. [4]

3 Exam-style questions

3.1 Let $g_n(x) = \sin(nx)/n$ on $\mathbb{R}$. Is convergence uniform? May its derivatives converge to the derivative of its limit? Use an explicit point to decide. [4]

Solutions

2.1 C. Bolzano–Weierstrass gives a convergent subsequence. The sequence $(-1)^n$ rules out the other necessary claims. The bounded sequence $(-1)^n$ is neither convergent, monotone nor Cauchy and does not tend to zero. Its even subsequence does converge.

2.2 C. The supremum of the absolute value is $1/n$, independent of x , which tends to zero. A, D and E fail because the uniform error bound includes every real x , including zero and noninteger inputs. B fails because the same bound $1/n$ tends to zero independently of x .

2.3 On $[0, 1]$, the limit is zero for $x < 1$ and one at 1. This limit is discontinuous, so continuous f_n cannot converge uniformly. More directly the supremum error on $[0, 1]$ is one. On $[0, a]$ the supremum is $a^n \rightarrow 0$, so convergence is uniform to zero. The p -series with $p=2$ converges absolutely. The alternating harmonic series converges by decreasing terms tending to zero, but the absolute series diverges; its convergence is conditional.

3.1 $\sup_{x \in \mathbb{R}} |g_n(x)| = 1/n \rightarrow 0$, so the functions converge uniformly to zero. But $g'_n(x) = \cos(nx)$, and at $x = 0$ these derivatives are all 1. The derivative of the zero limit is 0. Thus uniform convergence of the functions alone does not justify differentiating a limit; additional derivative-convergence hypotheses are needed.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.