

# 10.1 Rings, ideals and modules

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Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

Total: 10 marks

KEY WORDS ideal • torsion

## Objective

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Distinguish units, zero divisors and integral domains; Identify ideals and interpret quotient rings; Compare modules over rings with vector spaces over fields; Derive Boolean-ring properties without assuming commutativity.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

## 1 Key knowledge and worked examples

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**ideal** : An additive subgroup of a ring absorbing multiplication by all ring elements.

**torsion** : An element is annihilated by a nonzero scalar in the stated module.

For the domain, ideal and quotient examples below, use a commutative ring with identity 1 distinct from 0. A source problem may specify a general ring instead; do not assume its multiplication commutes unless stated or proved. A unit has a multiplicative inverse. A nonzero zero divisor multiplies some nonzero element to zero. An integral domain has no such zero divisors; cancellation of a nonzero factor then works. A field is a domain in which every nonzero element is a unit.  $\mathbb{Z}$  is a domain but not a field;  $\mathbb{Z}/6\mathbb{Z}$  is neither.

In  $\mathbb{Z}/6\mathbb{Z}$ ,  $2 \cdot 3 = 0$ , so 2 and 3 are zero divisors. The units are 1 and 5 because their gcd with 6 is 1. In  $\mathbb{Z}$ , the ideal  $5\mathbb{Z}$  gives the field  $\mathbb{Z}/5\mathbb{Z}$ , while  $6\mathbb{Z}$  gives a quotient with zero divisors. As  $\mathbb{Z}$ -modules, the map from  $\mathbb{Z}$  to  $\mathbb{Z}/6\mathbb{Z}$  has kernel  $6\mathbb{Z}$ ; the quotient identifies integers differing by a multiple of 6, rather than producing a real vector space.

## Multiplication in $\mathbb{Z}/6\mathbb{Z}$ : units and zero divisors

| Residue | Product mod 6                     | Classification        |
|---------|-----------------------------------|-----------------------|
| 1, 5    | $1 \cdot 1 = 1$ ; $5 \cdot 5 = 1$ | Units                 |
| 2, 3, 4 | $2 \cdot 3 = 0$ ; $4 \cdot 3 = 0$ | Nonzero zero divisors |
| 0       | $0 \cdot a = 0$                   | Zero                  |

*Original course illustration; its values belong to the worked example, not the later practice.*

## 2 Practice

2.1 How many units are in  $\mathbb{Z}/10\mathbb{Z}$ ?

[1]

- A 2
- B 3
- C 4
- D 5
- E 9

2.2 Which integer ideal is maximal in  $\mathbb{Z}$ ?

[1]

- A  $0\mathbb{Z}$
- B  $4\mathbb{Z}$
- C  $6\mathbb{Z}$
- D  $7\mathbb{Z}$
- E  $9\mathbb{Z}$

**2.3** In  $\mathbb{Z}/12\mathbb{Z}$ , list the units and nonzero zero divisors. Is the ideal  $(4)$  prime or maximal?<sup>[4]</sup>

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### 3 Exam-style questions

**3.1** As a  $\mathbb{Z}$ -module, can  $\mathbb{Z}/6\mathbb{Z}$  be free? Compare the ideals  $(2)$  and  $(x)$  in  $\mathbb{Z}$  and  $\mathbb{Q}[x]$ , respectively, using their quotients. <sup>[4]</sup>

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## Solutions

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**2.1 C.** Only 1, 3, 7, 9 are coprime to 10; each has a multiplicative inverse. There are four units. Two or three omit units; five or nine include a residue divisible by two or five.

**2.2 D.**  $\mathbb{Z}/7\mathbb{Z}$  is a field because 7 is prime. The composite quotients have zero divisors, and  $\mathbb{Z}/0\mathbb{Z}$  is  $\mathbb{Z}$ , not a field. The zero ideal has quotient  $\mathbb{Z}$ , which is not a field. Quotients by four, six and nine have nonzero zero divisors, so those ideals are not maximal.

**2.3** Units are the residues coprime to 12: 1, 5, 7, 11. The remaining nonzero residues 2, 3, 4, 6, 8, 9, 10 are zero divisors. For each such  $a$ ,  $a \cdot (12/\gcd(a, 12)) = 0$  modulo 12, with the second factor nonzero. The ideal  $(4) = \{0, 4, 8\}$  gives quotient isomorphic to  $\mathbb{Z}/4\mathbb{Z}$ . This is not a domain, since  $2 \cdot 2 = 0$ , and is not a field. Hence the ideal is neither prime nor maximal.

**3.1** A nonzero free  $\mathbb{Z}$ -module has no nonzero vector annihilated by a nonzero integer: examine each integer coordinate. But  $6[1] = 0$  in  $\mathbb{Z}/6\mathbb{Z}$ . It is not free; a field-style basis argument is invalid.  $\mathbb{Z}/(2) \cong \mathbb{F}_2$  is a field, so  $(2)$  is maximal and prime. Evaluation at zero maps  $\mathbb{Q}[x]$  onto  $\mathbb{Q}$  with kernel  $(x)$ ; its quotient is a field too. Both arguments use proper ideals in commutative rings with identity.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.