

Exercise walk-through

9.1

Groups, cosets and quotient maps

GRE · GRE Subject Test · GRE Mathematics

6. Choose and justify a method

Choose and justify a method

Before computing an order, check closure, associativity, an identity and an inverse for every element under the stated operation. A subset can inherit associativity yet fail closure or omit the identity. In a finite group, the order of an element is the least positive power giving the identity. In the additive group $\mathbb{Z}/n\mathbb{Z}$, it is the least positive multiple giving zero; the order of residue a is $n/\gcd(a,n)$. Lagrange's theorem says subgroup orders divide the group order.

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Quotient multiplication is well defined only when H is normal. All subgroups of an abelian group are normal. In a nonabelian group test $gHg^{-1}=H$; a subgroup of index two is normal. For permutations compose in the stated convention, here rightmost first. Disjoint cycle lengths give the permutation order by their least common multiple. Conjugation $h\sigma h^{-1}$ relabels the elements in σ 's cycles, so it preserves cycle lengths; conversely permutations with the same cycle lengths can be related by a relabelling.

18. Reasoning and check

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Conjugation by (123) sends (12) to (23) , which is outside H . Thus H is not normal. Left cosets still partition S_3 , but multiplication of those cosets is not well defined independently of representatives. The disjoint cycles in $(123)(45)$ have lengths three and two, so its order is $\text{lcm}(3, 2) = 6$. Element order and subgroup index are different quantities.

Exercise 2.1

2.1 What is the order of 8 in the additive group $\mathbb{Z}/20\mathbb{Z}$?

[1]

- **A** 2
- **B** 4
- **C** 5
- **D** 8
- **E** 20

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2.1 C. The least positive k with $8k$ divisible by 20 is 5; equivalently $20/\gcd(8,20)=5$. Two, four and eight multiples of eight are not zero modulo twenty; twenty works but is not the least positive count.

Exercises 2.2·2.3

2.2 A homomorphism from a finite group of order 30 has kernel of order 5. How many elements are in its image? [1]

- A 5
- B 6
- C 10
- D 25
- E 30

2.3 In the additive group $\mathbb{Z}/18\mathbb{Z}$, let $H = \langle 6 \rangle$. List H and its cosets. Determine the order of 6 and identify the quotient. [4]

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- D 25
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2.2 B. Fibres are kernel cosets of size 5, so the image has $30/5=6$ elements regardless of the codomain size. Five is the kernel size; ten, twenty-five and thirty do not equal the number of five-element fibres in thirty elements.

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2.3 $H = \{0, 6, 12\}$ has size three. The six cosets are $r + H = \{r, r + 6, r + 12\}$ for $r = 0, 1, 2, 3, 4, 5$. They partition all 18 residues. The element 6 has order three, while the quotient has order six. Reduction modulo six is onto with kernel H , so $(\mathbb{Z}/18\mathbb{Z})/H \cong \mathbb{Z}/6\mathbb{Z}$. Normality holds because the original group is abelian.

Exercise 3.1

3.1 In S_3 , let $H = \{e, (12)\}$. Check whether H is normal, and decide whether coset multiplication defines a quotient group. Separately find the order of $(123)(45)$ in S_5 . [4]

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Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.