

Exercise walk-through

7.1

Complex analysis and residues

GRE · GRE Subject Test · GRE Mathematics

6. Choose and justify a method

Choose and justify a method

Write $f(z)=u(x,y)+iv(x,y)$. Complex differentiability imposes $u_x=v_y$ and $u_y=-v_x$; with continuous first partials locally, the Cauchy–Riemann equations establish analyticity there. They differ from real differentiability of a two-coordinate map. The conjugate function $x-iy$ fails these equations on every open neighbourhood, although real partial derivatives exist. State the region being checked, not only a convenient point.

7. Choose and justify a method

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An analytic function is complex differentiable throughout a neighbourhood and has a local convergent power series. A removable singularity can be filled analytically when the function is bounded near the missing point. A pole has a finite principal part in its Laurent series; an essential singularity has infinitely many negative-power terms. The residue is the coefficient of $(z-a)^{-1}$, not necessarily the leading or largest negative-power term.

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For a positively oriented contour enclosing isolated singularities, the residue theorem gives $\oint f(z)dz = 2\pi i$ times the sum of enclosed residues, provided the function is analytic on the contour and elsewhere in the required interior. Reversing orientation changes the sign. Cauchy's derivative formula is $\oint g(z)/(z-a)^{m+1}dz = 2\pi i g^{(m)}(a)/m!$ under its analytic-domain hypotheses. A singularity on the contour prevents direct application; a singularity outside contributes nothing to this contour.

Exercise 2.1

2.1 The residue of $3/(z-1)$ at $z=1$ is:

[1]

- **A** 1
- **B** 3
- **C** 0
- **D** $2\pi i$
- **E** $6\pi i$

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2.1 B. The coefficient of the simple-pole term is 3. One is the pole location, not its coefficient. Zero loses the pole term; $2\pi i$ and $6\pi i$ are integral-like quantities, not the Laurent coefficient three.

Exercises 2.2·2.3

2.2 A positively oriented circle enclosing only that pole gives integral: [1]

- A 0
- B $3\pi i$
- C $6\pi i$
- D 3
- E $-6\pi i$

2.3 Use the Cauchy–Riemann equations to compare z^2 and $\bar{z}$. Find the residue of e^z/z^2 at zero and its integral around the counterclockwise unit circle. [4]

Exercises 2.2·2.3

2.2 A positively oriented circle enclosing only that pole gives integral: [1]

- A 0
- B $3\pi i$
- C $6\pi i$
- D 3
- E $-6\pi i$

2.2 C. Multiply residue 3 by $2\pi i$. Zero omits the pole; $3\pi i$ misses the factor two; three is the residue; negative $6\pi i$ reverses orientation.

2.3 Use the Cauchy–Riemann equations to compare z^2 and $\bar{z}$. Find the residue of e^z/z^2 at zero and its integral around the counterclockwise unit circle. [4]

2.3 For z^2 , $u = x^2 - y^2$ and $v = 2xy$: $u_x = 2x = v_y$ and $u_y = -2y = -v_x$. The partials

Exercise 3.1

3.1 Integrate $1/[z(z-2)]$ around $|z| = 1$, first counterclockwise and then clockwise. Can the same residue-theorem argument be used on $|z| = 2$? [4]

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3.1 Only zero is inside the unit circle. Its residue is $\lim_{z \rightarrow 0} 1/(z-2) = -1/2$. Thus the counterclockwise integral is $-\pi i$ and the clockwise integral is πi . The pole at 2 is outside and contributes nothing. The circle $|z| = 2$ passes through a pole. The ordinary contour integral is not defined by this formula; the theorem requires analyticity on the contour. An indentation or principal-value prescription would be a different, explicitly specified problem.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.