

Exercise walk-through

5.1

Real analysis and topology

GRE · GRE Subject Test · GRE Mathematics

Figure 11: Worked reasoning

$X = (0, 1)$: bounded, but incomplete

Cauchy sequence approaches a missing point



Check the labelled values and conditions.

16. Reasoning and check

Reasoning and check

X is bounded and closed relative to itself, since its complement in X is empty. The Cauchy sequence $1/(n+2)$ has no limit in X , so X is not complete and therefore not compact as a metric space. Directly, the relative open cover $\{(1/n, 1) : n \geq 2\}$ covers X but has no finite subcover. Heine–Borel requires closed and bounded as a subset of Euclidean space, not merely closed in an arbitrary chosen space. X is not closed in $\mathbb{R}$. No theorem hypothesis may change its ambient space halfway through the argument.

Exercises 2.1·2.2

2.1 Which subset of the real line is compact?

[1]

- **A** $(0,1)$
- **B** $[0,1]$
- **C** $[0,\text{infinity})$

2.2 A convergent real sequence must be:

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- **A** bounded
- **B** strictly increasing
- **C** made of integers

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- **A** $(0,1)$
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2.1 B. It is both closed and bounded in the real line. A omits its limit endpoints; C is unbounded. Both fail a Euclidean compactness condition.

2.2 A convergent real sequence must be:

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- **A** bounded
- **B** strictly increasing
- **C** made of integers

2.2 A. Convergence gives an eventual bound plus finitely many earlier terms. B fails for a constant or oscillating convergent sequence; C fails for the constant sequence one-half.

Exercise 2.3

2.3 Prove continuity of $f(x) = x^2$ at 2 using an explicit epsilon–delta choice. Explain

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2.3 Given $\epsilon > 0$, choose $\delta = \min(1, \epsilon/5)$. If $|x - 2| < \delta$, then $1 < x < 3$, so $|x + 2| < 5$.

$$|x^2 - 4| = |x - 2||x + 2| < 5\delta \leq \epsilon.$$

The choice depends on epsilon, not on the later input x . The interval $[1, 3]$ is nonempty and compact in the ordinary real metric. Continuity gives attained extrema. Here $x^2 \geq 1$, with equality at 1, so the minimum is 1.

Exercise 3.1

3.1 In the metric space $X = (0, 1)$ with ordinary distance, is X bounded? Is it closed in itself? Is it compact or complete? Explain why “closed and bounded implies compact” cannot be used here.

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3.1 X is bounded and closed relative to itself, since its complement in X is empty. The Cauchy sequence $1/(n + 2)$ has no limit in X , so X is not complete and therefore not compact as a metric space. Directly, the relative open cover $\{(1/n, 1) : n \geq 2\}$ covers X but has no finite subcover. Heine–Borel requires closed and bounded as a subset of Euclidean space, not merely closed in an arbitrary chosen space. X is not closed in $\mathbb{R}$. No theorem hypothesis may change its ambient space halfway through the argument.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.