

28. A.7 · LU factorisation and nullity of composed maps

Starter Quiz ·

GRE Subject Test · GRE Mathematics

Name: _____ Score: _____

1. Let $L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = (2, 7)$. Solve $LUx = b$; give x_1 .

2. Let $A, B : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ have nullities 1 and 2, respectively. What is the largest possible nullity of AB ?

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4. Some invertible square matrices require a row exchange before a factorisation with unit-lower L and upper-triangular U exists.
☐ True ☐ False
5. Solve $Ly = d$ for $L = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$ and $d = (4, 7)$. Give y_2 .

6. For $A = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$, $b = (4, 7)$ and the row-swap matrix P , which pair gives Pb and the solution x ?
☐ $Pb = (7, 4)$ and $x = (2, 1)$
☐ $Pb = (4, 7)$ and $x = (1, 2)$
☐ $Pb = (7, 4)$ and $x = (1, 2)$
☐ $Pb = (4, 7)$ and $x = (-13/2, 7/2)$
7. For $U = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, how many solutions do $Ux = (2, 0)$ and $Ux = (2, 1)$ have, in that order?
☐ Infinitely many; one
☐ None; infinitely many
☐ One; none
☐ Infinitely many; none
8. Using standard dense LU, what is the total arithmetic cost for k right sides of the same invertible $n \times n$ matrix, reusing its factors?
☐ $O(n^3 + kn^2)$

☐ $O(kn^3)$ with no possible saving

☐ $O(n^3)$ for arbitrarily large k

☐ $O(n^2 + k)$

9. For linear maps $B : V \rightarrow W$ and $A : W \rightarrow Z$, suppose $\text{rank } B = 4$ and $\text{rank } AB = 2$. Give $\dim(\text{im } B \cap \ker A)$.

10. Two square matrices always give the same nullity for AB and BA .

☐ True ☐ False

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Answers ·

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1. **0.5**

Forward substitution gives $y = (2, 1)$. Then $x_2 = 1$ and $x_1 = (2 - 1)/2 = 1/2$. The product $LU = \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix}$ checks the original system.

2. **3**

Rank $B = 3$. The image of B can contain the one-dimensional kernel of A , giving nullity $AB = 2 + 1 = 3$.

3. **2**

The intersection can have dimension $\max(0, 3 + 1 - 5) = 0$. Thus the minimum is $2 + 0 = 2$, attained by the two diagonal maps in the lesson.

4. **True**

True. For $A = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$, an unpivoted unit-lower factorisation would force $a_{21} = 0$. A row swap gives $PA = U$ and $L = I$.

5. **1**

Divide by $l_{11} = 2$: $y_1 = 2$. Then $y_2 = 7 - 3(2) = 1$. Assuming $y_1 = d_1$ would be wrong for this non-unit L .

6. $Pb = (7, 4)$ **and** $x = (1, 2)$

Swap the right side with the rows. Back substitution in $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} x = (7, 4)$ gives $x_2 = 2$, then $x_1 = 1$.

7. **Infinitely many; none**

The first system leaves x_2 free. The second requires $0 = 1$. A zero pivot does not allow a unique-solution division step.

8. $O(n^3 + kn^2)$

Factor once at cubic cost, then perform two quadratic-cost triangular solves for each right side. This is a general dense estimate.

9. **2**

Rank $AB = r - t$, so $t = 4 - 2 = 2$.

10. **False**

False. With $A = \text{diag}(1, 0)$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $AB = B$ has nullity 1 but $BA = 0$ has nullity 2.