

16. C.5 · Improper integrals and geometric applications

Starter Quiz ·

GRE Subject Test · GRE Mathematics

Name: _____ Score: _____

1. Evaluate $\int_1^\infty x^{-3} dx$.

2. Rotate $y = \sqrt{x}$, $0 \leq x \leq 4$, about the x -axis. The volume is $k\pi$. Find k .

3. Since $\lim_{\varepsilon \rightarrow 0} \left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^1 \right) \frac{dx}{x} = 0$, the improper integral $\int_{-1}^1 \frac{dx}{x}$ converges to 0.
☐ True ☐ False
4. Which integral converges?
☐ $\int_1^\infty \frac{dx}{x^2 + \sin^2 x}$
☐ $\int_1^\infty \frac{dx}{x}$
☐ $\int_1^\infty \frac{dx}{\sqrt{x}}$
☐ $\int_0^1 \frac{dx}{x^2}$
5. There is a real p for which $\int_0^\infty x^{-p} dx$ converges.
☐ True ☐ False
6. Evaluate $\int_e^\infty dx/[x(\ln x)^2]$.

7. For eventually positive f and g , a limit $f/g \rightarrow 0$ proves that their improper integrals have the same convergence behaviour.
☐ True ☐ False
8. The integral $\int_1^\infty (\sin x)/x dx$ converges absolutely.
☐ True ☐ False
9. What is the geometric area between $y = x$ and $y = x^2$ on $[-1, 1]$?

10. Rotate $x \leq y \leq 2$, $0 \leq x \leq 1$, about the x -axis. The volume is $k\pi$. What is k ?
☐ 1/3
☐ 7/3
☐ 11/3

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Answers ·

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1. **0.5**

$$\left[-\frac{1}{2x^2}\right]_1^\infty = \frac{1}{2}.$$

2. **8**

$$\pi \int_0^4 x \, dx = 8\pi.$$

3. **False**

Each half must converge on its own.

4. $\int_1^\infty \frac{dx}{x^2 + \sin^2 x}$

For $x \geq 1$, $0 < 1/(x^2 + \sin^2 x) \leq 1/x^2$, and $\int_1^\infty dx/x^2$ converges, so the comparison test gives convergence. $1/\sqrt{x}$ and $1/x$ decay too slowly at infinity, and $1/x^2$ blows up too fast at 0.

5. **False**

The piece near zero needs $p < 1$, while the tail at infinity needs $p > 1$. At $p = 1$, both pieces diverge logarithmically.

6. **1**

With $u = \ln x$, this is $\int_1^\infty u^{-2} \, du = 1$.

7. **False**

Take $f = 1/x^2$ and $g = 1/x$ on $[1, \infty)$. Their ratio tends to zero, but only the integral of f converges. A finite positive ratio gives the two-way limit comparison.

8. **False**

Integration by parts proves ordinary convergence. But on successive central pieces of sine half-waves, the absolute integral has harmonic lower bounds and diverges. Convergence is conditional.

9. **1**

Split at zero, where the upper curve changes. The two positive areas are $5/6$ and $1/6$, totalling 1. The signed integral $-2/3$ is not this area.

10. **11/3**

Each washer has outer radius 2 and inner radius x . Integrate $4 - x^2$ from 0 to 1, obtaining $11/3$. Squaring $2 - x$ would incorrectly square the thickness.