14. T.5 · Open sets, compactness and connectedness

Starter Quiz ·

GRE Subject Test · GRE Mathematics Name: _____ Score: _____

1. The closure of $\mathbb{Q}$ in $\mathbb{R}$ is

   ☐ $\mathbb{R}$

   ☐ $\varnothing$

   ☐ $\mathbb{R} \setminus \mathbb{Q}$

   ☐ $\mathbb{Q}$

2. The boundary of $(0,1) \cup (1,2)$ in $\mathbb{R}$ is

   ☐ $\{1\}$

   ☐ $[0,2]$

   ☐ $\{0,2\}$

   ☐ $\{0,1,2\}$

3. Which subsets of $\mathbb{R}$ are connected?

   ☐ $\{5\}$

   ☐ $[0,\infty)$

   ☐ $\mathbb{Q}$

   ☐ $[0,1] \cup [2,3]$

   ☐ $(0,1)$

   Tick every correct option.

4. Every compact subset of $\mathbb{R}$ is connected.

   ☐ True ☐ False

5. In $X = [0,2]$, the set $[0,1)$ is open because it equals X intersected with an _____ subset of $\mathbb{R}$.

   _____

6. For $A = [0,1/2)$ inside $X = [0,1]$, what is the status of 0 relative to A in X?

   ☐ Outside the closure.

   ☐ A boundary point.

   ☐ Not a point of X.

   ☐ An interior point.

7. The set $\{0\} \cup \{1/n : n \geq 1\}$ is compact in $\mathbb{R}$.

   ☐ True ☐ False

8. Why does the cover $U_n = (1/n, 1)$, $n \geq 2$, of $(0, 1)$ have no finite subcover?

   ☐ The intervals fail to cover any positive point.

   ☐ The set (0, 1) is unbounded.

   ☐ Any infinite cover automatically has no finite subcover.

   ☐ A finite choice with largest index N misses $1/(2N)$.

9. With distance 1 between distinct points, $\mathbb{N}$ is compact because it is complete and bounded.

   ☐ True ☐ False

10. A continuous real-valued function on a nonempty compact domain must

    ☐ be constant.

    ☐ have a connected domain.

    ☐ have an interval as its image.

    ☐ attain both a minimum and a maximum.

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Answers ·

1. $\mathbb{R}$

   Every real number is a limit of rationals.

2. $\{0, 1, 2\}$

   Closure [0, 2] minus interior (0,1) ∪ (1,2).

3. $\{5\}$; $[0, \infty)$; $(0, 1)$

   The connected subsets of $\mathbb{R}$ are intervals, including single points.

4. **False**

   False. {0, 1} is closed and bounded, hence compact, but it is not connected: the connected subsets of $\mathbb{R}$ are intervals.

5. **open**

   For example $(-1, 1)$.

6. **An interior point.**

   A sufficiently small ball in X around 0 is $[0, \varepsilon)$, lying inside A. Thus 0 is an interior point in X. In $\mathbb{R}$, however, every neighbourhood also contains negative points, so 0 is a boundary point there.

7. **True**

   It includes its only new accumulation point, 0, so it is closed in $\mathbb{R}$. It is also bounded, so Heine–Borel applies. Deleting 0 would destroy closedness and compactness.

8. **A finite choice with largest index N misses 1/(2N).**

   The sets are nested as n increases. A finite choice has a largest index N and union (1/N, 1), missing the point 1/(2N) of (0, 1). A cover can be infinite and still have a finite subcover, so the number of sets alone is not a proof.

9. **False**

   It is complete and bounded, but not totally bounded. Balls of radius 1/2 are singletons, and no finite set of them covers $\mathbb{N}$. Completeness plus ordinary boundedness is insufficient.

10. **attain both a minimum and a maximum.**

    Its image is a nonempty compact subset of $\mathbb{R}$, so its minimum and maximum belong to that image. The image need not be an interval unless connectedness is also assumed.