13. T.4 · Sequences, series and uniform convergence

Starter Quiz ·

GRE Subject Test · GRE Mathematics Name: _____ Score: _____

1. On $[0, \frac{1}{2}]$, what is $\sup |x^3 - 0|$?

   _____

2. Which sequence converges uniformly on $[0, 1]$?

   ☐ $x^{1/n}$

   ☐ $nx(1-x)^n$

   ☐ x^n

   ☐ x^n/n

3. For $\sum n/2^n$, the ratio test gives

   ☐ limit 0, inconclusive

   ☐ limit 1, inconclusive

   ☐ limit 2, so it diverges

   ☐ limit 1/2, so the series converges

4. A uniform limit of continuous functions is continuous.

   ☐ True ☐ False

5. $\sum(-1)^n/n$ converges, but not absolutely: it converges _____.

   _____

6. Every Cauchy sequence of rational numbers converges to a rational number.

   ☐ True ☐ False

7. If $a_n \to 0$, then $\sum_{n=1}^{\infty} a_n$ must converge.

   ☐ True ☐ False

8. At which $x \geq 0$ does $5xe^{-5x}$ attain its maximum?

   _____

9. Which series is uniformly convergent on $[-1, 1]$ by the M-test with a summable bound?

   ☐ $\sum_{n=1}^{\infty} nx^n$

   ☐ $\sum_{n=1}^{\infty} x^n/n$

   ☐ $\sum_{n=1}^{\infty} x^n/n^2$

□ $\sum_{n=1}^{\infty} x^n$

10. A triangular h_n has base $2/n$ and height n on $[0, 1]$, for $n \geq 2$. What is $\int_0^1 h_n(x)\,dx$?

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Answers ·

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1. **0.125**

   x^3 increases on [0, ½], so the supremum of $|x^3|$ is at x = ½: $(½)^3 = 1/8 = 0.125$.

2. x^n/n

   The sequence x^n/n has supremum error $1/n$, so converges uniformly to zero. For x^n the error supremum is 1. For $nx(1-x)^n$ the maximum is $(n/(n+1))^{n+1}$, tending to $1/e$ rather than zero. For $x^{1/n}$, the pointwise limit is 0 at $x = 0$ and 1 for $x > 0$, so the discontinuous limit rules out uniform convergence of these continuous functions.

3. **limit 1/2, so the series converges**

   $\frac{n+1}{2n} \to \frac{1}{2}$.

4. **True**

   True. If continuous $f_n \to f$ uniformly, an $\varepsilon/3$ argument shows f is continuous. Pointwise limits need not be: x^n on [0, 1] has a discontinuous limit.

5. **conditionally**

   Its absolute series is harmonic.

6. **False**

   Rational decimal truncations of $\sqrt{2}$ are Cauchy, but their unique real limit is irrational. The equivalence between Cauchy and convergent depends on completeness of the ambient space.

7. **False**

   The terms 1/n tend to zero, but the harmonic series diverges. Tending to zero is necessary, not sufficient.

8. **0.2**

   The derivative is $5e\hat{}(-5x)(1 - 5x)$, positive before x = 1/5 and negative after it. The maximum occurs at x = 1/5 = 0.2.

9. $\sum_{n=1}^{\infty} x^n/n^2$

   For the first series, $|x^n/n^2| \leq 1/n^2$ and $\sum 1/n^2$ converges. Each other series diverges at x = 1, so cannot converge uniformly on the whole interval.

10. **1**

    The area is half the base times the height: (1/2)(2/n)n = 1 for every n. Pointwise convergence to zero does not make these integrals tend to zero.