GRE Subject Test · GRE Mathematics Name: _____ Score: _____

1. Which polynomial is irreducible over $\mathbb{F}_2$?

   ☐ $x^2 + x$

   ☐ $x^2 + x + 1$

   ☐ x^2

   ☐ $x^2 + 1$

2. p is irreducible of degree 3 over $\mathbb{F}_5$. How many elements does $\mathbb{F}_5[x]/(p)$ have?

   _____

3. What is $[\mathbb{Q}(\sqrt[3]{2}, \sqrt{2}) : \mathbb{Q}]$?

   _____

4. A degree-4 polynomial over $\mathbb{Q}$ with no rational root can still be reducible over $\mathbb{Q}$.

   ☐ True ☐ False

5. In $\mathbb{F}_4 = \mathbb{F}_2[\alpha]$ with $\alpha^2 = \alpha + 1$, the inverse of α has the form $\alpha + c$. Enter $c \in \mathbb{F}_2$.

   _____

6. A quartic over a field is always irreducible if it has no root in that field.

   ☐ True ☐ False

7. Why is $\mathbb{F}_2[x]/(x^2)$ not a field, although it has four elements?

   ☐ It has characteristic 4.

   ☐ It contains a nonzero element whose square is zero.

   ☐ The class of 1 equals the class of 0.

   ☐ It has too few elements to be a field.

8. What is the characteristic of the field $\mathbb{F}_4$?

   _____

9. In $\mathbb{F}_4$ with $\alpha^2 = \alpha + 1$, what is α^8?

   ☐ $\alpha + 1$

   ☐ α

   ☐ 1

   ☐ 0

10. Find $[\mathbb{Q}(\sqrt{2}, \sqrt{8}) : \mathbb{Q}]$.

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1. $x^2 + x + 1$

   The quadratic $x^2 + x + 1$ takes value 1 at both elements 0 and 1 of $\mathbb{F}_2$, so has no root and is irreducible. The other choices factor as $(x + 1)^2$, $x(x + 1)$, and $x \cdot x$.

2. **125**

   An irreducible p of degree 3 makes $\mathbb{F}_5[x]/(p)$ a field whose elements are the remainders $a + bx + cx^2$ with a, b, $c \in \mathbb{F}_5$: $5^3 = 125$ elements.

3. **6**

   The cubic $x^3 - 2$ and quadratic $x^2 - 2$ are irreducible over $\mathbb{Q}$. The total degree is therefore divisible by 3 and by 2, hence by 6. Adjoining $\sqrt{2}$ to $\mathbb{Q}(\sqrt[3]{2})$ has degree at most 2, so the total degree is at most $3 \cdot 2 = 6$. The matching bounds force degree 6.

4. **True**

   True. It may factor into two quadratics with no rational roots, such as $(x^2 + 1)(x^2 + 2)$.

5. **1**

   $\alpha(\alpha + 1) = 1$.

6. **False**

   A quartic can factor as two quadratics without a linear factor. For example, $(x^2 + 1)(x^2 + 2)$ has no real root but is reducible over $\mathbb{R}$.

7. **It contains a nonzero element whose square is zero.**

   The nonzero class t of x satisfies $t^2 = 0$, so t is a zero divisor and cannot have an inverse. The count of four remainders does not establish the field property.

8. **2**

   The coefficients come from $\mathbb{F}_2$, so $1 + 1 = 0$. The characteristic is 2; the number of elements is 4.

9. $\alpha + 1$

   The multiplicative order of α is 3, so $\alpha^8 = \alpha^2 = \alpha + 1$. This is a product of eight factors, not a sum of eight copies.

10. **2**

    Since $\sqrt{8} = 2\sqrt{2}$, adjoining $\sqrt{8}$ adds no new elements after adjoining $\sqrt{2}$. The field is $\mathbb{Q}(\sqrt{2})$, of degree 2.