4. A.2 · Abstract algebra and number theory

Starter Quiz ·

GRE Subject Test · GRE Mathematics Name: _____ Score: _____

1. Which is a group?

   ☐ the integers under multiplication

   ☐ the nonzero rationals under multiplication

   ☐ the integers mod 8 without 0, under multiplication mod 8

   ☐ the positive integers under addition

2. Which residue has a multiplicative inverse modulo 9?

   ☐ 3

   ☐ 12

   ☐ 6

   ☐ 4

3. Find the least non-negative x with $5x \equiv 3 \pmod 7$.

   _____

4. $\mathbb{Z}_7$ is an integral domain.

   ☐ True ☐ False

5. A homomorphism is injective exactly when its _____ is trivial.

   _____

6. What is the order of the element 4 in the additive group $\mathbb{Z}_{12}$?

   _____

7. For reduction $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_4$, how many elements are in the kernel?

   _____

8. Which ring is an integral domain under the convention that 1 and 0 are distinct?

   ☐ The zero ring

   ☐ $\mathbb{Z}_4$

   ☐ $\mathbb{Q}[x]$

   ☐ $\mathbb{Z}_6$

9. Find the least nonnegative inverse of 17 modulo 43.

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10. How many distinct solution classes modulo 15 satisfy $6x \equiv 9 \pmod{15}$?

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1. **the nonzero rationals under multiplication**

   The others lack inverses (or closure).

2. **4**

   $\gcd(4, 9) = 1$; $4 \cdot 7 = 28 \equiv 1$.

3. **2**

   $5 \cdot 2 = 10 \equiv 3$.

4. **True**

   For n $\geq$ 2, the ring $\mathbb{Z}_n$ has no nonzero zero divisors exactly when n is prime. Since 7 is prime, $\mathbb{Z}_7$ is a field and hence an integral domain.

5. **kernel**

   Only the identity maps to the identity.

6. **3**

   The successive multiples are 4, 8, 0 modulo 12. The least positive number of additions returning to 0 is 3.

7. **3**

   The inputs sent to 0 modulo 4 are 0, 4 and 8. These are three distinct residues modulo 12.

8. $\mathbb{Q}[x]$

   In $\mathbb{Q}[x]$, the product of nonzero polynomials has nonzero leading coefficient, so there are no zero divisors. In $\mathbb{Z}_6$, $2 \cdot 3 = 0$; in $\mathbb{Z}_4$, $2 \cdot 2 = 0$. The zero ring has $1 = 0$.

9. **38**

   Bézout gives $1 = 2 \cdot 43 - 5 \cdot 17$. Thus -5, or 38 modulo 43, is the inverse. Direct check: $17 \cdot 38 = 646 = 15 \cdot 43 + 1$.

10. **3**

    The gcd is 3, which divides 9. Dividing both sides and the modulus gives $2x \equiv 3 \pmod 5$, so $x \equiv 4 \pmod 5$. The three classes modulo 15 are 4, 9 and 14.