

9.1 Groups, cosets and quotient maps

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS coset • normal subgroup

Objective

Verify group structure and compute element orders and subgroup indices; Use kernels and images to identify quotient groups; Distinguish normal subgroups from arbitrary subgroups; Classify permutation conjugacy by cycle type.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

coset : A translate of a subgroup that forms one part of the coset partition.

normal subgroup : A subgroup invariant under conjugation by every group element.

Before computing an order, check closure, associativity, an identity and an inverse for every element under the stated operation. A subset can inherit associativity yet fail closure or omit the identity. In a finite group, the order of an element is the least positive power giving the identity. In the additive group $\mathbb{Z}/n\mathbb{Z}$, it is the least positive multiple giving zero; the order of residue a is $n/\gcd(a,n)$. Lagrange's theorem says subgroup orders divide the group order. The converse is not a general existence theorem, and the order of a group is not the order of each element.

Define ϕ from $\mathbb{Z}/12\mathbb{Z}$ to $\mathbb{Z}/3\mathbb{Z}$ by reducing residues modulo 3. It is onto and preserves addition. Its kernel H is $\{0,3,6,9\}$, so $|H|=4$ and the index is $12/4=3$. The other cosets are $\{1,4,7,10\}$ and $\{2,5,8,11\}$. Thus $(\mathbb{Z}/12\mathbb{Z})/H$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The element 3 in the original group has order $12/\gcd(3,12)=4$, not 3.

Reduction modulo 3: three fibres, four elements each

Quotient class	Elements of $\mathbb{Z}/12\mathbb{Z}$	Image in $\mathbb{Z}/3\mathbb{Z}$
H	0, 3, 6, 9	0
$1 + H$	1, 4, 7, 10	1
$2 + H$	2, 5, 8, 11	2

Reduction mod 3 splits $\mathbb{Z}/12\mathbb{Z}$ into three fibres of four elements each

2 Practice

2.1 What is the order of 8 in the additive group $\mathbb{Z}/20\mathbb{Z}$?

[1]

- A 2
 - B 4
 - C 5
 - D 8
 - E 20
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2.2 A homomorphism from a finite group of order 30 has kernel of order 5. How many elements are in its image?

[1]

- A 5
 - B 6
 - C 10
 - D 25
 - E 30
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2.3 In the additive group $\mathbb{Z}/18\mathbb{Z}$, let $H = \langle 6 \rangle$. List H and its cosets. Determine the order of 6 and identify the quotient. [4]

3 Exam-style questions

3.1 In S_3 , let $H = \{e, (12)\}$. Check whether H is normal, and decide whether coset multiplication defines a quotient group. Separately find the order of $(123)(45)$ in S_5 . [4]
