

4.1 Abstract algebra and number theory

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS homomorphism • field

Objective

Use groups, subgroups, homomorphisms and quotient structures; Distinguish rings, integral domains and fields; Apply divisibility, congruences and elementary number theory.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

homomorphism : A map preserving the relevant operation.

field : A commutative ring where each nonzero element has an inverse.

A group needs closure, associativity, identity and inverses. Commutativity is an additional condition.

Modulo 8, 3 has inverse 3 because $3 \cdot 3 \equiv 1$. But 2 has no inverse because $\gcd(2,8)=2$. Modulo a prime, every nonzero residue has an inverse; this gives a finite field.

Same addition rule; different invertibility

Ring	Multiplicative units	Check
mod 5	1, 2, 3, 4	every nonzero residue
mod 6	1, 5	2 times 3 is zero

Mod 5 every nonzero residue is invertible; mod 6 the zero divisors 2 and 3 are not

2 Practice

2.1 The integers under addition form: [1]

- A a group
- B a field under integer division
- C a set without identity

2.2 Which has an inverse modulo 9? [1]

- A 3
- B 6
- C 2

2.3 Compare the additive groups and multiplicative structures of $\mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$. List all multiplicative units in each. Explain why removing zero does not always leave a multiplicative group. [4]

3 Exam-style questions

3.1 The reduction map $\phi : \mathbb{Z} \rightarrow \mathbb{Z}/6\mathbb{Z}$ is given by $\phi(n) = [n]$. Determine its kernel, image and quotient. Solve $2x \equiv 2 \pmod{6}$ completely and explain why cancelling 2 modulo 6 fails. [4]
