

31.1 Metric completeness and closure from a basis

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS complete metric space • neighbourhood basis

Objective

Check a pullback metric using the properties of its defining map; Decide completeness through the image of an isometry; Determine closure using every basic neighbourhood instead of Euclidean intuition.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

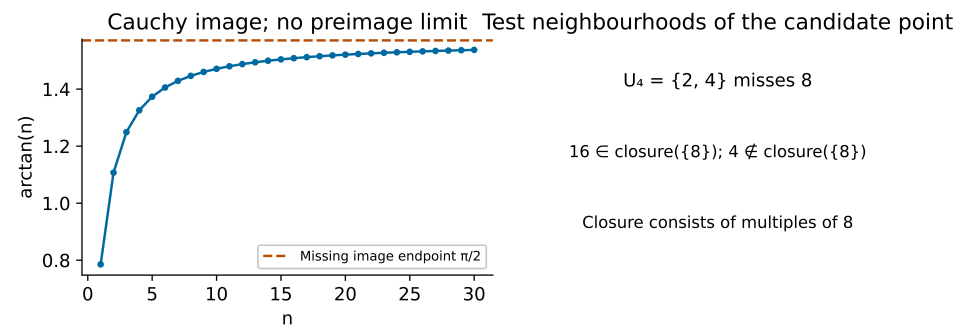
1 Key knowledge and worked examples

complete metric space : A metric space in which every Cauchy sequence converges to a point of that space.

neighbourhood basis : Basic neighbourhoods sufficient to test local topological properties.

For an injective map $h: X \rightarrow \mathbb{R}$, $d(x, y) = |h(x) - h(y)|$ is a metric: symmetry and the triangle inequality come from $\mathbb{R}$, and injectivity ensures $d(x, y) = 0$ only when $x = y$. The map h is an isometry onto its image $h(X)$. If h is not injective, this construction may be only a pseudometric. Bounded distances do not prove a space complete, and a familiar set of points can have very different Cauchy behaviour under different metrics.

Under $d(x, y) = |\arctan x - \arctan y|$, the integer sequence goes toward a missing image endpoint rather than a point of $\mathbb{R}$, so it is Cauchy without convergence in that space. In the divisor basis, 16 lies in closure of $\{8\}$, since any basic set containing 16 also contains 8. But 4 does not: $U_4 = \{2, 4\}$ is a neighbourhood of 4 missing 8. This explicitly separates multiples from divisors.



The image of a Cauchy sequence can converge to a point outside the image

2 Practice

2.1 Which statement holds for $d(x, y) = |\arctan x - \arctan y|$ on $\mathbb{R}$? [1]

- A It is not a metric
- B It is complete because $\arctan$ is continuous
- C It is incomplete because its isometric image omits its endpoints
- D Every unbounded sequence fails to be Cauchy
- E Its distance equals ordinary absolute difference

2.2 For the divisor basis in the lesson, which point belongs to closure of $\{8\}$? [1]

- A 2
- B 3
- C 4
- D 6
- E 16

2.3 On $\mathbb{R}$ compare $d_1(x, y) = |x^3 - y^3|$ and $d_2(x, y) = |\arctan x - \arctan y|$. Prove they are metrics and decide completeness. [4]

3 Exam-style questions

3.1 On $X = \{2, 3, 4, \dots\}$ use the divisor basis $U_k = \{n \in X : n \mid k\}$. Find the closure of $\{6\}$ and the interior of $\{6\}$. Verify the basis conditions, including empty intersections. [4]
