31.1 Metric completeness and closure from a basis

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS complete metric space • neighbourhood basis

Objective

Check a pullback metric using the properties of its defining map; Decide completeness through the image of an isometry; Determine closure using every basic neighbourhood instead of Euclidean intuition.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

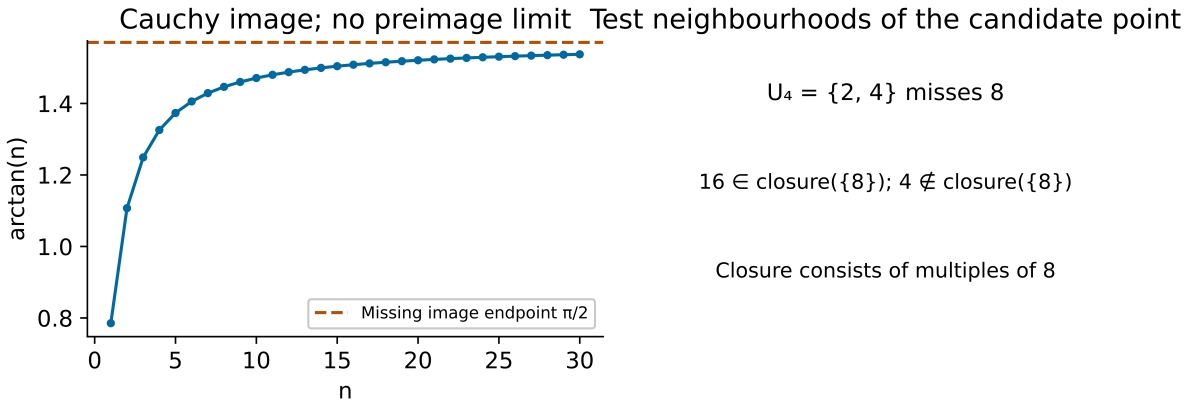
1 Key knowledge and worked examples

complete metric space : A metric space in which every Cauchy sequence converges to a point of that space.

neighbourhood basis : Basic neighbourhoods sufficient to test local topological properties.

For an injective map h:X→R, d(x,y)=|h(x)−h(y)| is a metric: symmetry and the triangle inequality come from R, and injectivity ensures d(x,y)=0 only when x=y. The map h is an isometry onto its image h(X). If h is not injective, this construction may be only a pseudometric. Bounded distances do not prove a space complete, and a familiar set of points can have very different Cauchy behaviour under different metrics.

Under d(x,y)=|arctan x−arctan y|, the integer sequence goes toward a missing image endpoint rather than a point of R, so it is Cauchy without convergence in that space. In the divisor basis, 16 lies in closure of {8}, since any basic set containing 16 also contains 8. But 4 does not: U_4={2,4} is a neighbourhood of 4 missing 8. This explicitly separates multiples from divisors.



The image of a Cauchy sequence can converge to a point outside the image

2 Practice

2.1 Which statement holds for d(x,y)=|arctan x−arctan y| on R? [1]

- **A** It is not a metric
- **B** It is complete because arctan is continuous
- **C** It is incomplete because its isometric image omits its endpoints
- **D** Every unbounded sequence fails to be Cauchy
- **E** Its distance equals ordinary absolute difference

2.2 For the divisor basis in the lesson, which point belongs to closure of {8}? [1]

- **A** 2
- **B** 3
- **C** 4
- **D** 6
- **E** 16

2.3 On $\mathbb{R}$ compare $d_1(x,y) = |x^3 - y^3|$ and $d_2(x,y) = |\arctan x - \arctan y|$. Prove they are metrics and decide completeness. [4]

3 Exam-style questions

3.1 On $X = \{2, 3, 4, \dots\}$ use the divisor basis $U_k = \{n \in X : n \mid k\}$. Find the closure of $\{6\}$ and the interior of $\{6\}$. Verify the basis conditions, including empty intersections. [4]