

30.1 Related rates and removable quotient limits

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS related rate • removable limit

Objective

Translate a geometric rate into a derivative of the relevant quantity; Derive and differentiate a spherical-cap volume formula; Separate continuous quotient extension from differentiable extension.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

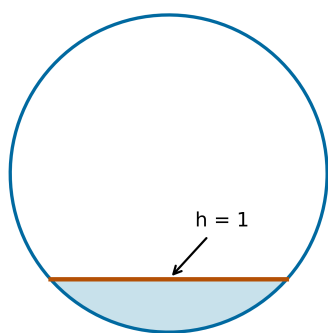
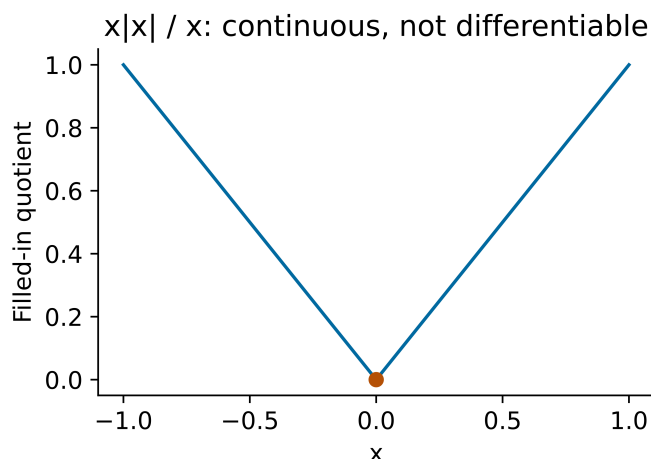
related rate : A rate obtained by differentiating the relationship between changing quantities.

removable limit : A finite nearby limit used to fill in a missing function value continuously.

For a quantity V depending on a changing depth $h(t)$, the chain rule gives $dV/dt = V'(h) dh/dt$. A draining tank has $dh/dt < 0$, so signed dV/dt is negative; an outflow magnitude is its negative. Draw the geometry and name the instantaneous depth before substituting numbers. Do not treat the rate as a static volume divided by elapsed time unless the situation actually specifies a constant average rate.

For a sphere with $R=3$ and depth $h=1$, cross-sectional area is $\pi(6-1)=5\pi$. A depth decrease of $1/4$ per time unit therefore removes volume at magnitude $5\pi/4$ per time unit. For $f(x)=\sin(2x)$, $g(x)=3x$, the quotient extends at zero with value $2/3$. This limit says nothing about the size of a tank and should not be substituted as a derivative rate without identifying the dependent quantities.

$R = 3$; section area at $h = 1$ is 5π



The cross-section at $h = 1$ has $\text{radius}^2 = 9 - 2^2 = 5$, so area 5π

2 Practice

2.1 A radius-3 spherical tank has depth 1 decreasing at $1/4$ per time unit. What is the positive outflow magnitude? [1]

- **A** $\pi/4$
- **B** $3\pi/4$
- **C** $5\pi/4$
- **D** $9\pi/4$
- **E** 36π

2.2 For $x \neq 0$, $f(x) = x|x|$, $g(x) = x$. What happens when f/g is filled in at zero? [1]

- **A** It stays undefined
- **B** It becomes discontinuous
- **C** It is continuous but not differentiable
- **D** It is differentiable with derivative 1
- **E** It is constantly zero

2.3 A spherical tank has radius 2 m and water depth h measured from its bottom. Derive $V(h)$. At $h=1$ m and $dh/dt = -0.1$ m/min, find signed volume change and outflow magnitude. [4]

3 Exam-style questions

3.1 Suppose f, g are continuously differentiable near zero, $f(0) = g(0) = 0$, $f'(0) = 4$, $g'(0) = 2$. Find the removable limit of $(f^2 - f)/(2g - g^3)$. Do these assumptions imply differentiability of the extended quotient? Give a counterexample to that general implication. [4]
