3.1 Linear algebra

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS rank • eigenvalue

Objective

Relate rank, nullity and solutions of linear systems; Analyse vector spaces and linear transformations; Compute eigenvalues, determinants and diagonalisation conditions.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

rank : Dimension of the image of a linear map.

eigenvalue : A scalar satisfying $Av=\lambda v$ for a nonzero v.

A real vector space is closed under its specified addition and scalar multiplication and satisfies the vector-space axioms. A basis is an independent spanning list: every vector has a unique coordinate representation in that list. To test independence of matrix columns, solve the homogeneous system $Ac=0$; only the zero coefficient vector means independence. A spanning set can contain redundant vectors, so spanning alone does not make a basis.

$A=[[1,1],[0,1]]$ has characteristic polynomial $(1-\lambda)^2$. Eigenvectors satisfy $y=0$, so its eigenspace has dimension one. Two independent eigenvectors are needed to diagonalise a 2×2 matrix; A is not diagonalizable.

A repeated eigenvalue can have different eigenspace dimensions

Matrix	Eigenvalue	Dimension	Diagonalizable?
[[1, 1], [0, 1]]	1, twice	1	No
[[1, 0], [0, 1]]	1, twice	2	Yes

A repeated eigenvalue gives a diagonalizable matrix only when its eigenspace has full dimension

2 Practice

2.1 A 3-column matrix has rank 2. Its nullity is: [1]

- **A** 1
- **B** 2
- **C** 3
- **D** 0
- **E** 5

2.2 det A=0 implies: [1]

- **A** A is invertible
- **B** A is singular
- **C** All entries are zero
- **D** Every column is zero
- **E** Every eigenvalue is nonzero

2.3 For $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $T(x, y, z) = (x + 2y + z, 2x + 4y + 2z)$, find bases of the kernel and image. Solve $T(v) = (3, 6)$ and $T(v) = (3, 7)$. [4]

3 Exam-style questions

3.1 Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Determine its eigenvalues, determinant and diagonalisation over $\mathbb{R}$ and $\mathbb{C}$. Find A^n for integers $n \geq 1$ without repeated multiplication. [4]