

28.1 LU factorisation and nullity of composed maps

Name: _____ Class: _____ Date: _____ Total: 10 marks

KEY WORDS forward substitution • nullity

Objective

Solve a factored linear system by forward and backward substitution; Apply row permutations consistently when pivoting is required; Bound the kernel dimension of a composition using image-kernel intersections.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

forward substitution : Solving a lower triangular system from its first equation downward.

nullity : The dimension of the kernel of a linear map.

A factorisation $A=LU$ expresses a square matrix as lower triangular L and upper triangular U . With unit diagonal L , forward substitution is especially simple. An invertible matrix need not admit this form without row exchanges: a zero leading pivot can require a permutation. For $PA=LU$, solve $Ly=Pb$, then $Ux=y$. The permutation acts on the right side as well as the coefficient matrix. Nonzero leading principal pivots justify the usual no-exchange elimination, not invertibility alone.

Let $L=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$, $U=\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, and $b=\begin{pmatrix} -1 \\ -1 \\ 12 \end{pmatrix}$. Forward substitution gives $y=\begin{pmatrix} -1 \\ 1 \\ 8 \end{pmatrix}$. Back substitution gives $x_3=2$, $x_2=(1-4)/3=-1$, and $x_1=(-1-(-1)+2)/2=1$. Therefore $x=\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Multiplying Ux gives y and multiplying Ly gives b , independently checking the factor order.

LU solve: forward through L , then backward through U

Stage	Known right side	Result
$Ly = b$	$b = \begin{pmatrix} -1 \\ -1 \\ 12 \end{pmatrix}$	$y = \begin{pmatrix} -1 \\ 1 \\ 8 \end{pmatrix}$
$Ux = y$	$y = \begin{pmatrix} -1 \\ 1 \\ 8 \end{pmatrix}$	$x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Solve with LU by forward substitution through L , then back substitution through U

2 Practice

- 2.1** For the stated unit-diagonal L and $b=\begin{pmatrix} -1 \\ -1 \\ 12 \end{pmatrix}$, what is the second coordinate of y solving $Ly=b$? [1]
- **A** -3
 - **B** -1
 - **C** 0
 - **D** 1
 - **E** 12
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- 2.2** A, B are maps $\mathbb{R}^6 \rightarrow \mathbb{R}^6$ with nullity $A=2$ and nullity $B=3$. What range is possible for nullity of $A \circ B$? [1]
- **A** 0 to 2
 - **B** 2 to 3
 - **C** 3 to 5
 - **D** 5 to 6
 - **E** Always 5

2.3 Let $L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $b = (5, 12)^T$. Solve $LUx=b$ and verify in the multiplied matrix. For $PA=LU$, where does P act when solving $Ax=b$? [4]

3 Exam-style questions

3.1 Let A and B be endomorphisms of $\mathbb{R}^5$ with nullities two and three, respectively. Find all possible nullities of AB, and justify both the bound and attainability. [4]
