27.1 Implicit differentiation and the inverse Jacobian

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS Jacobian matrix • local inverse

Objective

Differentiate a coupled implicit system as a linear system; Use a nonzero Jacobian determinant to justify a local inverse; Distinguish inverse partial derivatives from scalar reciprocal rules.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

Jacobian matrix : The matrix of first partial derivatives of a vector-valued map.

local inverse : An inverse defined on neighbourhoods of a particular input and output point.

For a differentiable map F(u,v)=(x,y)=(f(u,v),g(u,v)), its Jacobian is J=[[f_u,f_v],[g_u,g_v]]. Small changes satisfy [dx,dy] (transpose)=J[du,dv] (transpose) to first order. To find u_x while holding y fixed, differentiate both defining equations with respect to x: f_u u_x+f_v v_x=1 and g_u u_x+g_v v_x=0. These are a coupled linear system, not two independent scalar inverse rules.

Take x=u^2+v and y=u−v. At (u,v)=(1,0), the output is (1,1), and J=[[2,1],[1,−1]] has determinant −3. The inverse is [[1/3,1/3],[1/3,−2/3]], so u_x=1/3 and v_y=−2/3 there. Multiplying J by this inverse gives the identity. The reciprocal 1/f_u=1/2 is not u_x because changing u also requires changing v to keep y fixed.

Coupled sensitivity: inverse entries differ from reciprocals

Forward map	Jacobian	Inverse Jacobian
x = u + v; y = u + 2v	[[1, 1], [1, 2]]	[[2, −1], [−1, 1]]
x = u^2 + v; y = u − v at (u, v) = (1, 0)	[[2, 1], [1, −1]]	[[1/3, 1/3], [1/3, −2/3]]

Invert the whole Jacobian matrix; its entries are not the reciprocals of the forward entries

2 Practice

2.1 For x=u+v,y=u+2v, what is u_x when y is held fixed? [1]

- **A** 0
- **B** 1/2
- **C** 1
- **D** 2
- **E** −1

2.2 J=[[2,1],[1,−1]]. What is v_y in the inverse derivative matrix? [1]

- **A** −2/3
- **B** −1/2
- **C** 1/3
- **D** 1/2
- **E** 2

2.3 Let $x = u^2 + v, y = u - v$. At $(u, v) = (1, 0)$, find the inverse Jacobian and the four inverse partial derivatives. Check by multiplication. [4]

3 Exam-style questions

3.1 For $F(u, v) = (u^2 - v^2, 2uv)$, decide whether its derivative is invertible away from the origin and whether F is globally injective on the punctured plane. What does a zero determinant at the origin establish by itself? [4]