

26.1 Weak compositions, loop invariants and flowchart tracing

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS weak composition • loop invariant

Objective

Count indistinguishable allocations with nonnegative or positive constraints; Trace a flowchart in statement order with correct reset behaviour; Use a loop invariant and a progress measure to justify an algorithm.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

weak composition : An ordered allocation of a total into nonnegative integer parts.

loop invariant : A property preserved before and after each iteration of a loop.

The number of nonnegative integer solutions of $x_1 + \cdots + x_k = n$ is $C(n+k-1, k-1)$, for $n \geq 0$ and $k \geq 1$. Represent n identical stars separated into k named boxes by $k-1$ bars; adjacent bars and end bars allow zero entries. Boxes are distinguishable, objects are not. Counting distinct tokens would instead assign each token independently, giving k^n before other constraints. Decide the model from what the objects and recipients represent, not from a familiar-looking binomial coefficient.

Distribute 11 identical tokens among 4 named boxes. Nonnegative allocations number $C(14,3)=364$; all-positive allocations number $C(10,3)=120$. Therefore 244 allocations have at least one empty box. For the odd-sum test with $N=10$, S progresses 0,1,4,9,16; it overshoots and rejects 10. For $N=9$ it exits at $S=9$ after three additions. The next addends are 1,3,5,7,9; sum and next-addend columns must not be confused.

Odd-sum loop: $S = m^2$, $B = 2m + 1$ before the next addition

Iterations m	Sum S	Next addend B
0	0	1
1	1	3
2	4	5
3	9	7
4	16	9

The loop invariant $S = m^2$ holds before each addition

2 Practice

2.1 How many ways put 11 identical tokens into 4 named boxes, allowing empty boxes? [1]

- **A** 44
- **B** 120
- **C** 244
- **D** 364
- **E** 4^{11}

2.2 Starting from $S=0$, add successive odd numbers 1,3,5,... until $S \geq 10$. What is S at exit? [1]

- **A** 9
- **B** 10
- **C** 11
- **D** 16
- **E** 25

2.3 Count nonnegative solutions of $a + b + c = 8$ with $a \geq 2$. How many also have $b, c \geq 1$? How many of the first set have at least one of b, c equal to zero? [4]

3 Exam-style questions

3.1 Trace this algorithm for $N=15$ and $N=16$: initialise $S=0$, $k=1$; while $S < N$, execute $S=S+k$, then $k=k+2$; report whether $S=N$. State an invariant, and justify termination for integer $N \geq 0$. [4]
