17.1 Multivariable extrema and constrained optimisation

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS Hessian • Lagrange multiplier

Objective

Classify two-variable critical points with the Hessian; Use Lagrange multipliers with a regular constraint; Compare extrema and attainable values under regular constraints.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

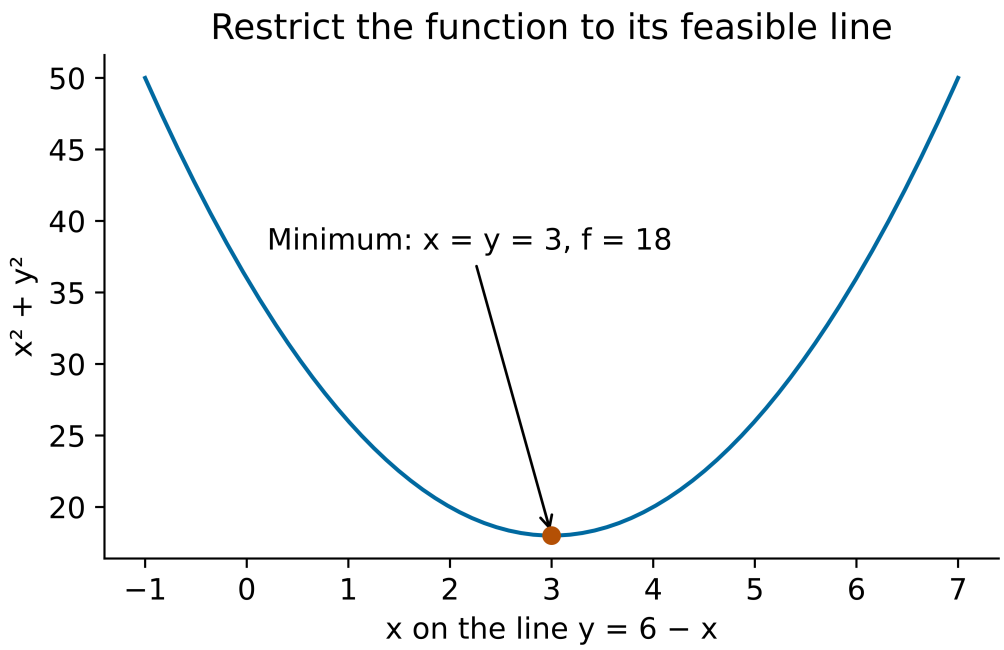
1 Key knowledge and worked examples

Hessian : The matrix of second partial derivatives used to study local curvature.

Lagrange multiplier : A scalar relating objective and regular constraint gradients at a constrained extremum.

For a differentiable function on an open domain, an interior local extremum has zero gradient. This condition is necessary, not sufficient. For two variables at a critical point, let $D=f_{xx}f_{yy}-(f_{xy})^2$. If $D>0$, $f_{xx}>0$ gives a strict local minimum and $f_{xx}<0$ gives a strict local maximum; $D<0$ gives a saddle. These tests require suitable second-derivative regularity near the point.

Minimise x^2+y^2 subject to $x+y=6$. The constraint gradient is $(1,1)$, so $2x=\lambda$ and $2y=\lambda$. Thus $x=y$ and the constraint gives $x=y=3$, with value 18. Completing the square on the line gives $x^2+(6-x)^2=2(x-3)^2+18$, proving a global minimum. There is no maximum on this unbounded line, despite the existence of a multiplier candidate.



Substitute the constraint first: the minimum is at $x = y = 3$, where $x^2 + y^2 = 18$

2 Practice

2.1 At a critical point f_xx=2, f_yy=−2 and f_xy=0. What does the second-derivative test give? [1]

- **A** Strict minimum
- **B** Strict maximum
- **C** Saddle point
- **D** Inconclusive
- **E** No critical point

2.2 What is the maximum of xy subject to $x^2+y^2=2$? [1]

- **A** -2
- **B** -1
- **C** 0
- **D** 1
- **E** 2

2.3 Classify the origin for $f(x,y) = x^2 - y^2$ and for $g(x,y) = x^4 + y^4$. State what a zero Hessian determinant permits you to conclude. [4]

3 Exam-style questions

3.1 Find all extrema of $f(x,y) = xy$ subject to $x^2 + y^2 = 2$. Compare with minimising $x^2 + y^2$ on the line $x + y = 4$, including whether maxima exist. [4]