

16.1 Improper integrals and geometric applications

Name: _____ Class: _____ Date: _____

Total: 10 marks**KEY WORDS** improper integral • principal value

Objective

Define improper integrals by limits at each singular boundary; Select area, volume and arc-length formulas from the geometry; Separate convergence of integrals from signed cancellation.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

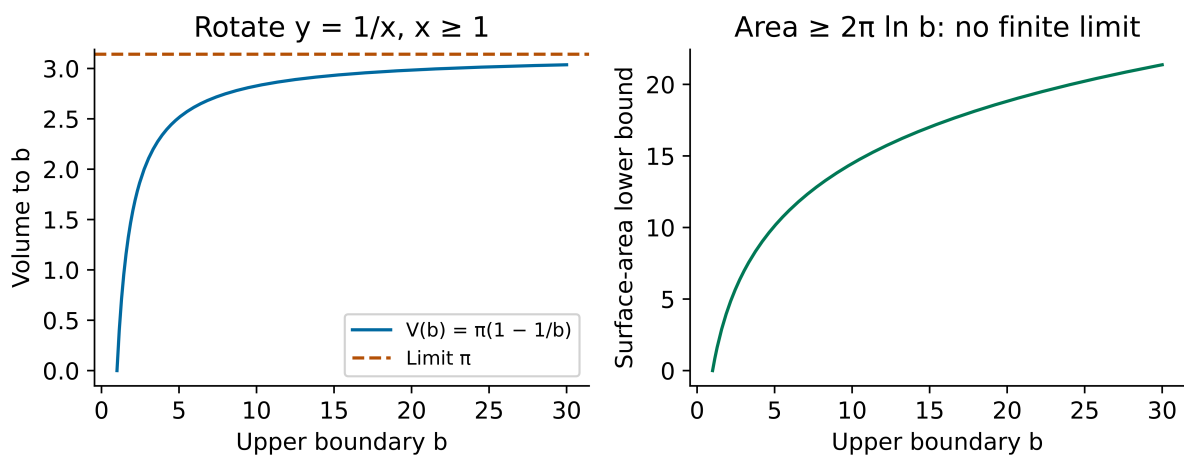
1 Key knowledge and worked examples

improper integral : An integral defined by limits at infinite or singular boundaries.

principal value : A limit using prescribed symmetric cancellation that may exist when an ordinary improper integral diverges.

An infinite integration endpoint is replaced by a finite bound and a limit. At a singular point inside the interval, split the integral and require both one-sided integrals to converge separately. Symmetric cancellation can define a Cauchy principal value but does not prove convergence of the ordinary improper integral. For $1/x$ across zero, the two sides diverge even though symmetric cutoffs cancel.

Rotate $y=1/x$ for $x \geq 1$ around the x -axis. The disk volume is π times the integral of $1/x^2$, so $V=\pi$. The lateral surface integrand is $2\pi(1/x)\sqrt{1+1/x^4}$, at least $2\pi/x$; its improper integral diverges. Finite volume therefore does not imply finite surface area. This comparison avoids trying to find an unnecessary antiderivative.



Gabriel's horn: finite volume π , infinite surface area

2 Practice

2.1 For which p does the integral of x^{-p} from zero to 1 converge?

[1]

- A $p > 1$
- B $p \geq 1$
- C $p < 1$
- D $p \leq 1$
- E All real p

2.2 Rotating the region under $y = x^2$ from $x = 0$ to $x = 1$ around the x -axis gives which volume?

[1]

- A $\pi/2$
- B $\pi/3$
- C $\pi/4$
- D $\pi/5$
- E $2\pi/3$

2.3 For which real p does $\int_0^\infty x^{-p} dx$ converge? Compare the ordinary improper integral of $1/x$ on $[-1, 1]$ with its symmetric principal value. [4]

3 Exam-style questions

3.1 Rotate the region under $y = 1/x$, $x \geq 1$, around the x-axis. Prove its volume is finite and its lateral surface area is infinite. [4]
