

13.1 Sequences, series and uniform convergence

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS Cauchy sequence • uniform convergence

Objective

Use Cauchy, monotone convergence and subsequence criteria; Distinguish pointwise from uniform convergence of functions; Check hypotheses before interchanging limits and integrals.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

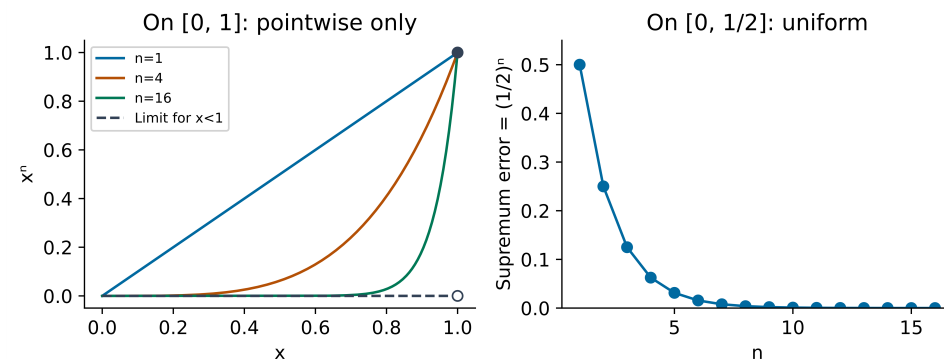
1 Key knowledge and worked examples

Cauchy sequence : A sequence whose sufficiently late terms are arbitrarily close to one another.

uniform convergence : Convergence with one error threshold index valid at every point of the domain.

A convergent real sequence is Cauchy, and every real Cauchy sequence converges because $\mathbb{R}$ is complete. In $\mathbb{Q}$ a Cauchy sequence can approach an irrational number and fail to converge within $\mathbb{Q}$. A bounded monotone real sequence converges. A bounded sequence need not converge, but Bolzano–Weierstrass guarantees a convergent subsequence; $(-1)^n$ has two different subsequential limits.

On $[0,1]$, $f_n(x)=x^n$ tends to 0 for $x<1$ and to 1 at $x=1$. The limit is discontinuous, so convergence cannot be uniform; directly, the supremum error is 1, approached below 1. On $[0,a]$ with $0\leq a<1$, the limit is zero and the supremum is a^n , which tends to zero, so convergence is uniform. Changing the domain changes the conclusion.



x^n converges only pointwise on $[0, 1]$, but uniformly on $[0, 1/2]$

2 Practice

2.1 Which statement must hold for a bounded real sequence? [1]

- A It converges
- B It is monotone
- C It has a convergent subsequence
- D It is Cauchy
- E Its terms tend to zero

2.2 For $f_n(x)=\sin(nx)/n$ on $\mathbb{R}$, convergence to zero is what type? [1]

- A Not pointwise
- B Pointwise but not uniform
- C Uniform
- D Only at integer x
- E Divergent at zero

2.3 Determine pointwise and uniform convergence of $f_n(x) = x^n$ on $[0, 1]$ and on $[0, a]$ for fixed $0 < a < 1$. Compare $\sum 1/n^2$ with the alternating harmonic series for absolute convergence. [4]

3 Exam-style questions

3.1 Let $g_n(x) = \sin(nx)/n$ on $\mathbb{R}$. Is convergence uniform? May its derivatives converge to the derivative of its limit? Use an explicit point to decide. [4]
