

11.1 Polynomials and field extensions

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS irreducible polynomial • extension degree

Objective

Test polynomial irreducibility over the specified field; Construct small finite fields from irreducible polynomials; Use extension degrees and the tower law; Use cyclotomic roots and coefficient relations to compute sums and products.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

irreducible polynomial : A positive-degree polynomial with no factorisation into smaller positive degrees over the stated field.

extension degree : The dimension of an extension field as a vector space over its base field.

A polynomial is irreducible over a field if it has positive degree and no factorisation into polynomials of smaller positive degrees there. A quadratic or cubic is irreducible exactly when it has no root in that field. This root test alone fails for degree four or higher; for instance $(x^2+1)(x^2+2)$ has no real root but is reducible over $\mathbb{R}$. Always name the base field.

In $\mathbb{F}_2[\alpha]$ with $\alpha^2+\alpha+1=0$, multiply $\alpha(\alpha+1) = \alpha^2 + \alpha = (\alpha+1) + \alpha = 1$. Thus $\alpha^{-1}=\alpha+1$. Also $\alpha^3=1$ and α is not 1, so its multiplicative order is 3. This produces a field of four elements. By contrast $2 \cdot 2=0$ in $\mathbb{Z}/4\mathbb{Z}$, proving that quotient is not a field.

$\mathbb{F}_2[\alpha]$, with $\alpha^2 = \alpha + 1$: all nonzero elements invertible

$\times$	1	α	$\alpha + 1$
1	1	α	$\alpha + 1$
α	α	$\alpha + 1$	1
$\alpha + 1$	$\alpha + 1$	1	α

Every nonzero element of $\mathbb{F}_2[\alpha]$ has an inverse, so it is a field with four elements

2 Practice

2.1 Which polynomial is irreducible over $\mathbb{F}_2$? [1]

- A x^2
- B x^2+1
- C x^2+x
- D x^2+x+1
- E x^3+x^2

2.2 If $[\mathbb{L}:\mathbb{K}]=3$ and $[\mathbb{M}:\mathbb{L}]=2$, what is $[\mathbb{M}:\mathbb{K}]$? [1]

- A 1
- B 2
- C 3
- D 5
- E 6

2.3 Construct $\mathbb{F}_2[x]/(x^2 + x + 1)$. List its elements and find the inverse of the residue $\alpha = [x]$. [4]

3 Exam-style questions

3.1 Is $x^4 + 4$ irreducible over $\mathbb{Q}$? Does the absence of rational roots answer that question? Can a degree-two extension of $\mathbb{Q}$ contain a root of an irreducible cubic over $\mathbb{Q}$? [4]
