

# 7.1 Complex analysis and residues

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_ Total: 10 marks

KEY WORDS analytic • residue

## Objective

Test complex differentiability; Use contour integrals and residues; Evaluate contour integrals with simple and higher-order pole residues.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

## 1 Key knowledge and worked examples

**analytic** : Complex differentiable throughout a neighbourhood.

**residue** : Coefficient of  $(z-a)^{-1}$  in a Laurent series.

Write  $f(z)=u(x,y)+iv(x,y)$ . Complex differentiability imposes  $u_x=v_y$  and  $u_y=-v_x$ ; with continuous first partials locally, the Cauchy–Riemann equations establish analyticity there. They differ from real differentiability of a two-coordinate map. The conjugate function  $x-iy$  fails these equations on every open neighbourhood, although real partial derivatives exist. State the region being checked, not only a convenient point.

For  $f(z)=\text{conjugate}(z)$ ,  $u=x$  and  $v=-y$ . Then  $u_x=1$  but  $v_y=-1$ , so  $f$  is not complex differentiable. For  $1/(z-2)$ , a positively oriented circle  $|z-2|=1$  encloses a simple pole of residue 1, hence the integral is  $2\pi i$ .

The residue is the coefficient of  $(z - 1)^{-1}$

| Function           | Pole order | Residue at 1 |
|--------------------|------------|--------------|
| $3/(z - 1)$        | 1          | 3            |
| $e^z/(z - 1)^2$    | 2          | e            |
| $e^{2z}/(z - 1)^3$ | 3          | $2e^2$       |

*The residue is the coefficient of  $(z - 1)^{-1}$  in the Laurent series*

## 2 Practice

2.1 The residue of  $3/(z-1)$  at  $z=1$  is: [1]

- A 1
- B 3
- C 0
- D  $2\pi i$
- E  $6\pi i$

2.2 A positively oriented circle enclosing only that pole gives integral: [1]

- A 0
- B  $3\pi i$
- C  $6\pi i$
- D 3
- E  $-6\pi i$

**2.3** Use the Cauchy–Riemann equations to compare  $z^2$  and  $\bar{z}$ . Find the residue of  $e^z/z^2$  at zero and its integral around the counterclockwise unit circle. [4]

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### 3 Exam-style questions

**3.1** Integrate  $1/[z(z-2)]$  around  $|z|=1$ , first counterclockwise and then clockwise. Can the same residue-theorem argument be used on  $|z|=2$ ? [4]

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## Solutions

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**2.1 B.** The coefficient of the simple-pole term is 3. One is the pole location, not its coefficient. Zero loses the pole term;  $2\pi i$  and  $6\pi i$  are integral-like quantities, not the Laurent coefficient three.

**2.2 C.** Multiply residue 3 by  $2\pi i$ . Zero omits the pole;  $3\pi i$  misses the factor two; three is the residue; negative  $6\pi i$  reverses orientation.

**2.3** For  $z^2$ ,  $u = x^2 - y^2$  and  $v = 2xy$ :  $u_x = 2x = v_y$  and  $u_y = -2y = -v_x$ . The partials are continuous everywhere, so the function is analytic everywhere. For  $\bar{z}$ ,  $u = x, v = -y$  and  $u_x = 1 \neq -1 = v_y$ ; it is nowhere complex differentiable. Since  $e^z = 1 + z + z^2/2 + \cdots$ , the coefficient of  $z^{-1}$  in  $e^z/z^2$  is 1. Therefore the residue is 1 and the integral is  $2\pi i$ . The double pole does not force a zero residue.

**3.1** Only zero is inside the unit circle. Its residue is  $\lim_{z \rightarrow 0} 1/(z - 2) = -1/2$ . Thus the counterclockwise integral is  $-\pi i$  and the clockwise integral is  $\pi i$ . The pole at 2 is outside and contributes nothing. The circle  $|z| = 2$  passes through a pole. The ordinary contour integral is not defined by this formula; the theorem requires analyticity on the contour. An indentation or principal-value prescription would be a different, explicitly specified problem.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.