

5.1 Real analysis and topology

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS compact • supremum

Objective

Apply sequence and function limit definitions; Distinguish compactness, connectedness and completeness; Use metric-space and elementary topological reasoning.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

compact : Every open cover has a finite subcover.

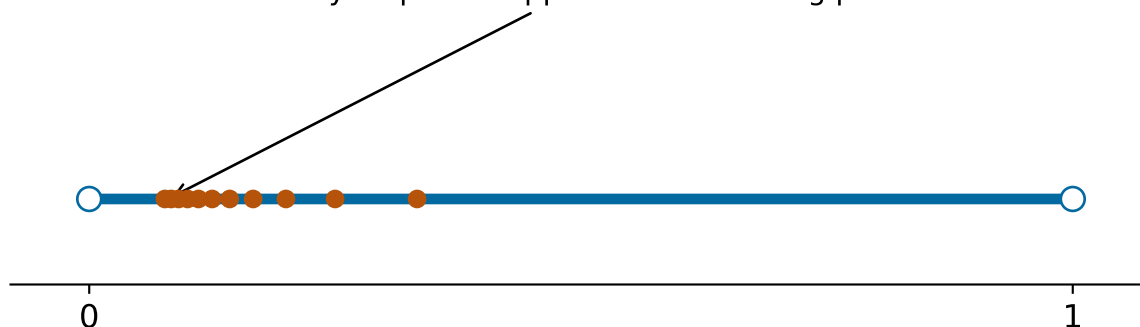
supremum : The least upper bound of a set.

An epsilon–delta statement controls all sufficiently close inputs. The quantifier order matters.

$f(x)=x$ on $(0,1)$ is continuous and bounded but never equals its supremum 1. On $[0,1]$, the same function attains its maximum at 1. The missing endpoint explains why the compact-domain theorem does not apply to the first case.

$X = (0, 1)$: bounded, but incomplete

Cauchy sequence approaches a missing point



$(0, 1)$ is bounded but not complete: a Cauchy sequence can approach a point outside it

2 Practice

2.1 Which subset of the real line is compact? [1]

- **A** $(0,1)$
 - **B** $[0,1]$
 - **C** $[0,\text{infinity})$
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2.2 A convergent real sequence must be: [1]

- **A** bounded
 - **B** strictly increasing
 - **C** made of integers
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2.3 Prove continuity of $f(x) = x^2$ at 2 using an explicit epsilon–delta choice. Explain separately why it attains a minimum on $[1,3]$. [4]

3 Exam-style questions

3.1 In the metric space $X = (0, 1)$ with ordinary distance, is X bounded? Is it closed in itself? Is it compact or complete? Explain why “closed and bounded implies compact” cannot be used here. [4]

Solutions

2.1 B. It is both closed and bounded in the real line. A omits its limit endpoints; C is unbounded. Both fail a Euclidean compactness condition.

2.2 A. Convergence gives an eventual bound plus finitely many earlier terms. B fails for a constant or oscillating convergent sequence; C fails for the constant sequence one-half.

2.3 Given $\epsilon > 0$, choose $\delta = \min(1, \epsilon/5)$. If $|x - 2| < \delta$, then $1 < x < 3$, so $|x + 2| < 5$.

$$|x^2 - 4| = |x - 2||x + 2| < 5\delta \leq \epsilon.$$

The choice depends on epsilon, not on the later input x . The interval $[1, 3]$ is nonempty and compact in the ordinary real metric. Continuity gives attained extrema. Here $x^2 \geq 1$, with equality at 1, so the minimum is 1.

3.1 X is bounded and closed relative to itself, since its complement in X is empty. The Cauchy sequence $1/(n + 2)$ has no limit in X , so X is not complete and therefore not compact as a metric space. Directly, the relative open cover $\{(1/n, 1) : n \geq 2\}$ covers X but has no finite subcover. Heine–Borel requires closed and bounded as a subset of Euclidean space, not merely closed in an arbitrary chosen space. X is not closed in $\mathbb{R}$. No theorem hypothesis may change its ambient space halfway through the argument.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.