31.1 Metric completeness and closure from a basis

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS complete metric space • neighbourhood basis

Objective

Check a pullback metric using the properties of its defining map; Decide completeness through the image of an isometry; Determine closure using every basic neighbourhood instead of Euclidean intuition.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

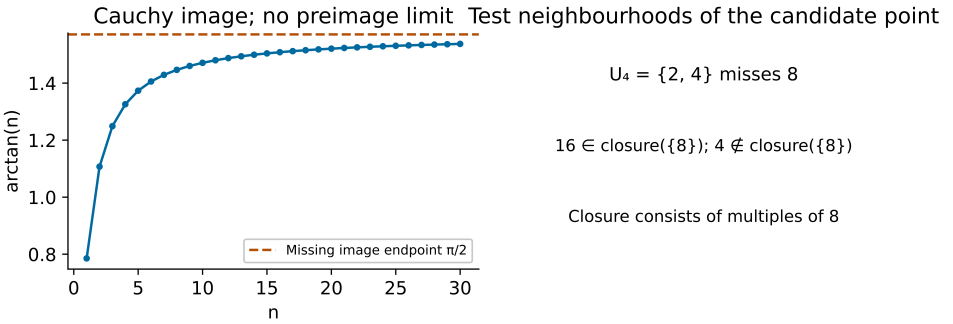
1 Key knowledge and worked examples

complete metric space : A metric space in which every Cauchy sequence converges to a point of that space.

neighbourhood basis : Basic neighbourhoods sufficient to test local topological properties.

For an injective map h:X→R, d(x,y)=|h(x)−h(y)| is a metric: symmetry and the triangle inequality come from R, and injectivity ensures d(x,y)=0 only when x=y. The map h is an isometry onto its image h(X). If h is not injective, this construction may be only a pseudometric. Bounded distances do not prove a space complete, and a familiar set of points can have very different Cauchy behaviour under different metrics.

Under d(x,y)=|arctan x−arctan y|, the integer sequence goes toward a missing image endpoint rather than a point of R, so it is Cauchy without convergence in that space. In the divisor basis, 16 lies in closure of {8}, since any basic set containing 16 also contains 8. But 4 does not: U_4={2,4} is a neighbourhood of 4 missing 8. This explicitly separates multiples from divisors.



The image of a Cauchy sequence can converge to a point outside the image

2 Practice

2.1 Which statement holds for d(x,y)=|arctan x−arctan y| on R? [1]

- **A** It is not a metric
- **B** It is complete because arctan is continuous
- **C** It is incomplete because its isometric image omits its endpoints
- **D** Every unbounded sequence fails to be Cauchy
- **E** Its distance equals ordinary absolute difference

2.2 For the divisor basis in the lesson, which point belongs to closure of {8}? [1]

- **A** 2
- **B** 3
- **C** 4
- **D** 6
- **E** 16

2.3 On $\mathbb{R}$ compare $d_1(x,y) = |x^3 - y^3|$ and $d_2(x,y) = |\arctan x - \arctan y|$. Prove they are metrics and decide completeness. [4]

3 Exam-style questions

3.1 On $X = \{2, 3, 4, \ldots\}$ use the divisor basis $U_k = \{n \in X : n \mid k\}$. Find the closure of $\{6\}$ and the interior of $\{6\}$. Verify the basis conditions, including empty intersections. [4]

Solutions

2.1 C. The injective map defines a metric. The image is an open bounded interval, and arctan n tends to a missing endpoint. Injectivity proves this is a metric. Continuity does not give completeness, unbounded n is Cauchy in this metric, and arctangent distance is not ordinary absolute distance.

2.2 E. Closure consists of multiples of 8. The point 16 has every neighbourhood containing 8; U_4 demonstrates why the divisor 4 does not belong. Closure requires multiples of eight. Two, three, four and six have divisor neighbourhoods missing eight; sixteen is a multiple and every neighbourhood contains eight.

2.3 Both defining maps are injective, giving positive separation; absolute distance gives symmetry and the triangle inequality. Each map is an isometry onto its real image. The cube map has image all of $\mathbb{R}$, a complete space, so d1 is complete. The arctangent image is the open interval $(-\pi/2, \pi/2)$, which is incomplete. The sequence n is d2-Cauchy because arctan n tends to pi/2, but no real point has that image, so it has no d2-limit. Bounded metric values do not imply completeness.

3.1 Every x belongs to Ux. An intersection is $U_{\gcd(k,l)}$ if the gcd is at least two, and is empty for gcd one. Thus a point in an intersection has a basic neighbourhood within it. The smallest basic neighbourhood of x is Ux, contained in every Uk that contains x. It meets $\{6\}$ exactly when $6 \mid x$. Therefore the closure is $\{6, 12, 18, \ldots\}$, the multiples of six. The point six is not interior to its singleton because U6 also contains two and three. No other point is in that singleton, so its interior is empty. This closure calculation uses all basic neighbourhoods, not Euclidean intuition.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.