

31.1 Metric completeness and closure from a basis

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS complete metric space • neighbourhood basis

Objective

Check a pullback metric using the properties of its defining map; Decide completeness through the image of an isometry; Determine closure using every basic neighbourhood instead of Euclidean intuition.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

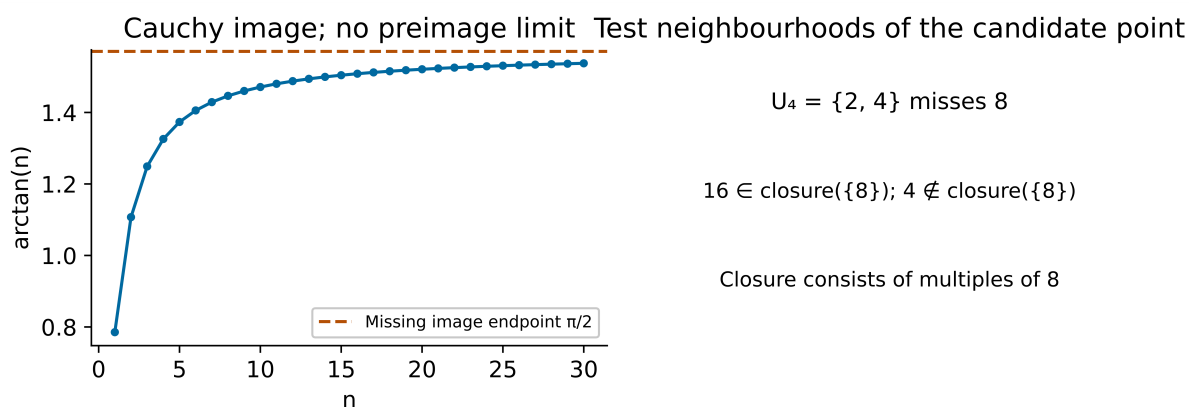
1 Key knowledge and worked examples

complete metric space : A metric space in which every Cauchy sequence converges to a point of that space.

neighbourhood basis : Basic neighbourhoods sufficient to test local topological properties.

For an injective map $h: X \rightarrow \mathbb{R}$, $d(x, y) = |h(x) - h(y)|$ is a metric: symmetry and the triangle inequality come from $\mathbb{R}$, and injectivity ensures $d(x, y) = 0$ only when $x = y$. The map h is an isometry onto its image $h(X)$. If h is not injective, this construction may be only a pseudometric. Bounded distances do not prove a space complete, and a familiar set of points can have very different Cauchy behaviour under different metrics.

Under $d(x, y) = |\arctan x - \arctan y|$, the integer sequence goes toward a missing image endpoint rather than a point of $\mathbb{R}$, so it is Cauchy without convergence in that space. In the divisor basis, 16 lies in closure of $\{8\}$, since any basic set containing 16 also contains 8. But 4 does not: $U_4 = \{2, 4\}$ is a neighbourhood of 4 missing 8. This explicitly separates multiples from divisors.



The image of a Cauchy sequence can converge to a point outside the image

2 Practice

2.1 Which statement holds for $d(x,y)=|\arctan x - \arctan y|$ on $\mathbb{R}$? [1]

- **A** It is not a metric
- **B** It is complete because arctan is continuous
- **C** It is incomplete because its isometric image omits its endpoints
- **D** Every unbounded sequence fails to be Cauchy
- **E** Its distance equals ordinary absolute difference

2.2 For the divisor basis in the lesson, which point belongs to closure of $\{8\}$? [1]

- **A** 2
- **B** 3
- **C** 4
- **D** 6
- **E** 16

2.3 On $\mathbb{R}$ compare $d_1(x,y) = |x^3 - y^3|$ and $d_2(x,y) = |\arctan x - \arctan y|$. Prove they are metrics and decide completeness. [4]

3 Exam-style questions

3.1 On $X = \{2, 3, 4, \dots\}$ use the divisor basis $U_k = \{n \in X : n \mid k\}$. Find the closure of $\{6\}$ and the interior of $\{6\}$. Verify the basis conditions, including empty intersections. [4]

Solutions

2.1 C. The injective map defines a metric. The image is an open bounded interval, and $\arctan n$ tends to a missing endpoint. Injectivity proves this is a metric. Continuity does not give completeness, unbounded n is Cauchy in this metric, and arctangent distance is not ordinary absolute distance.

2.2 E. Closure consists of multiples of 8. The point 16 has every neighbourhood containing 8; U_4 demonstrates why the divisor 4 does not belong. Closure requires multiples of eight. Two, three, four and six have divisor neighbourhoods missing eight; sixteen is a multiple and every neighbourhood contains eight.

2.3 Both defining maps are injective, giving positive separation; absolute distance gives symmetry and the triangle inequality. Each map is an isometry onto its real image. The cube map has image all of $\mathbb{R}$, a complete space, so d_1 is complete. The arctangent image is the open interval $(-\pi/2, \pi/2)$, which is incomplete. The sequence n is d_2 -Cauchy because $\arctan n$ tends to $\pi/2$, but no real point has that image, so it has no d_2 -limit. Bounded metric values do not imply completeness.

3.1 Every x belongs to U_x . An intersection is $U_{\gcd(k,l)}$ if the gcd is at least two, and is empty for gcd one. Thus a point in an intersection has a basic neighbourhood within it. The smallest basic neighbourhood of x is U_x , contained in every U_k that contains x . It meets $\{6\}$ exactly when $6 \mid x$. Therefore the closure is $\{6, 12, 18, \dots\}$, the multiples of six. The point six is not interior to its singleton because U_6 also contains two and three. No other point is in that singleton, so its interior is empty. This closure calculation uses all basic neighbourhoods, not Euclidean intuition.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.