30.1 Related rates and removable quotient limits

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS related rate • removable limit

Objective

Translate a geometric rate into a derivative of the relevant quantity; Derive and differentiate a spherical-cap volume formula; Separate continuous quotient extension from differentiable extension.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

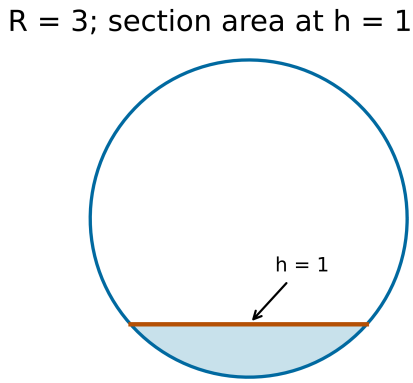
1 Key knowledge and worked examples

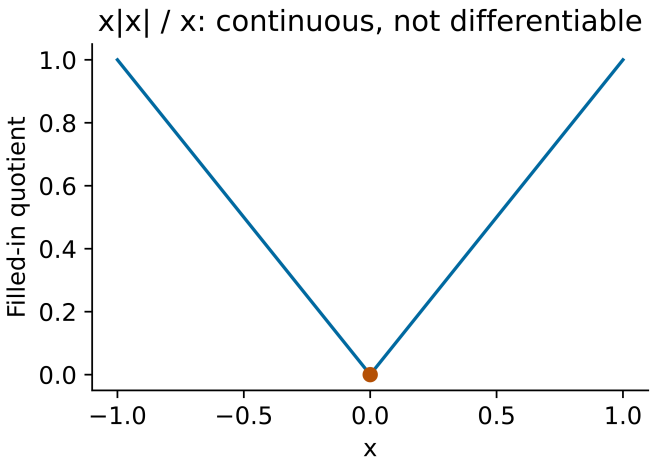
related rate : A rate obtained by differentiating the relationship between changing quantities.

removable limit : A finite nearby limit used to fill in a missing function value continuously.

For a quantity V depending on a changing depth h(t), the chain rule gives $dV/dt = V'(h)\,dh/dt$. A draining tank has $dh/dt < 0$, so signed dV/dt is negative; an outflow magnitude is its negative. Draw the geometry and name the instantaneous depth before substituting numbers. Do not treat the rate as a static volume divided by elapsed time unless the situation actually specifies a constant average rate.

For a sphere with $R=3$ and depth $h=1$, cross-sectional area is $\pi(6-1)=5\pi$. A depth decrease of $1/4$ per time unit therefore removes volume at magnitude $5\pi/4$ per time unit. For $f(x)=\sin(2x), g(x)=3x$, the quotient extends at zero with value $2/3$. This limit says nothing about the size of a tank and should not be substituted as a derivative rate without identifying the dependent quantities.





The cross-section at $h = 1$ has radius$^2 = 9 - 2^2 = 5$, so area 5π

2 Practice

2.1 A radius-3 spherical tank has depth 1 decreasing at 1/4 per time unit. What is the positive outflow magnitude? [1]

- **A** $\pi/4$
- **B** $3\pi/4$
- **C** $5\pi/4$
- **D** $9\pi/4$
- **E** 36π

2.2 For $x \neq 0$, $f(x)=x|x|, g(x)=x$. What happens when f/g is filled in at zero? [1]

- **A** It stays undefined
- **B** It becomes discontinuous
- **C** It is continuous but not differentiable
- **D** It is differentiable with derivative 1
- **E** It is constantly zero

2.3 A spherical tank has radius 2 m and water depth h measured from its bottom. Derive V(h). At h=1 m and $dh/dt = -0.1$ m/min, find signed volume change and outflow magnitude. [4]

3 Exam-style questions

3.1 Suppose f,g are continuously differentiable near zero, $f(0) = g(0) = 0$, $f'(0) = 4$, $g'(0) = 2$. Find the removable limit of $(f^2 - f)/(2g - g^3)$. Do these assumptions imply differentiability of the extended quotient? Give a counterexample to that general implication. [4]

Solutions

2.1 C. Cross-section area is $\pi(2\mathrm{Rh}-\mathrm{h}^2)=5\pi$. Multiply by depth-decrease magnitude 1/4. The section area is five pi and the speed magnitude one-quarter. The other coefficients use radius, whole-sphere quantities or omit the correct section factor.

2.2 C. The quotient is |x|. Filling in zero gives continuity, while the left and right derivatives are -1 and 1. Filling zero removes the missing value, so the extension is continuous. Its unequal one-sided slopes rule out differentiability with derivative one; it is not constant.

2.3 At depth s, the cross-sectional radius obeys $r(s)^2 = 2Rs - s^2$. Integrating disks gives $V(h) = \pi(Rh^2 - h^3/3)$ for $0 \le h \le 2R$.

$$\frac{dV}{dt} = \pi(2Rh - h^2)\frac{dh}{dt}.$$

$$\frac{dV}{dt} = \pi[2(2\text{ m})(1\text{ m}) - (1\text{ m})^2](-0.1\text{ m/min}) = -0.3\pi\text{ m}^3/\text{min}.$$

The outflow magnitude is 0.3π m^3/min. The sign describes loss of stored water, not negative physical outflow.

3.1 Since g'(0) is nonzero, the denominator is nonzero near zero except at zero. Factor the expression as $(f/g)(f-1)/(2-g^2)$. Its limit is $(4/2)(-1)/2 = -1$. First-order differentiability gives $f/g \to f'(0)/g'(0)$; it does not give second-order control. For a counterexample preserving these derivatives, take $f(x) = 4x + x|x|$ and $g(x) = 2x$. They are continuously differentiable. For nonzero x the quotient becomes $(2 + |x|/2)(-1 + 4x + x|x|)/(2 - 4x^2)$. Its expansion is $-1 + 4x - |x|/4 + O(x^2)$, giving different one-sided derivatives. Thus the extension is continuous but not differentiable.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.