

3.1 Linear algebra

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS rank • eigenvalue

Objective

Relate rank, nullity and solutions of linear systems; Analyse vector spaces and linear transformations; Compute eigenvalues, determinants and diagonalisation conditions.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

rank : Dimension of the image of a linear map.

eigenvalue : A scalar satisfying $Av=\lambda v$ for a nonzero v .

A real vector space is closed under its specified addition and scalar multiplication and satisfies the vector-space axioms. A basis is an independent spanning list: every vector has a unique coordinate representation in that list. To test independence of matrix columns, solve the homogeneous system $Ac=0$; only the zero coefficient vector means independence. A spanning set can contain redundant vectors, so spanning alone does not make a basis.

$A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ has characteristic polynomial $(1-\lambda)^2$. Eigenvectors satisfy $y=0$, so its eigenspace has dimension one. Two independent eigenvectors are needed to diagonalise a 2×2 matrix; A is not diagonalizable.

A repeated eigenvalue can have different eigenspace dimensions

Matrix	Eigenvalue	Dimension	Diagonalizable?
$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	1, twice	1	No
$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	1, twice	2	Yes

A repeated eigenvalue gives a diagonalizable matrix only when its eigenspace has full dimension

2 Practice

2.1 A 3-column matrix has rank 2. Its nullity is: [1]

- **A** 1
- **B** 2
- **C** 3
- **D** 0
- **E** 5

2.2 $\det A=0$ implies: [1]

- **A** A is invertible
- **B** A is singular
- **C** All entries are zero
- **D** Every column is zero
- **E** Every eigenvalue is nonzero

2.3 For $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with $T(x, y, z) = (x + 2y + z, 2x + 4y + 2z)$, find bases of the kernel and image. Solve $T(v) = (3, 6)$ and $T(v) = (3, 7)$. [4]

3 Exam-style questions

3.1 Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Determine its eigenvalues, determinant and diagonalisation over $\mathbb{R}$ and $\mathbb{C}$. Find A^n for integers $n \geq 1$ without repeated multiplication. [4]

Solutions

2.1 A. Rank-nullity gives $3-2=1$. B, C and E exceed the one free column allowed by rank two; D would require three independent columns.

2.2 B. A zero determinant means the square matrix is singular. It can have many nonzero entries; a single dependence is enough. A and E contradict singularity. C and D are stronger than dependence and fail, for example, for $\text{diag}(1,0)$.

2.3 The second row is twice the first, so rank is one. The homogeneous equation gives $x = -2y - z$.

$$\ker T = \text{span}\{(-2, 1, 0), (-1, 0, 1)\}.$$

The two displayed vectors are independent. Every output is $(s, 2s)$, and $(1, 2)$ occurs; this is an image basis. Rank plus nullity is $1 + 2 = 3$, the domain dimension. For $(3, 6)$, all solutions are $(3 - 2a - b, a, b)$, with $a, b \in \mathbb{R}$. For $(3, 7)$ there is no solution, since $7 \neq 2 \cdot 3$.

3.1 The characteristic polynomial is $(2 - \lambda)^2$ and $\det A = 4$. Solving $(A - 2I)v = 0$ gives $v_2 = 0$, so the eigenspace has dimension one over either field. Two independent eigenvectors are required; the matrix is invertible but not diagonalizable over either field. Write $A = 2I + N$ where $N^2 = 0$. The commuting binomial expansion leaves only two terms:

$$A^n = 2^n I + n2^{n-1}N = \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}.$$

Multiplying this expression by A gives the formula at $n + 1$, independently checking the coefficient.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.