29.1 Integration by parts, order reversal and symmetry

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS integration by parts • reflection symmetry

Objective

Derive integration-by-parts reductions with valid endpoint limits; Reverse a double integral by reconstructing its region; Use reflection symmetry while checking removable endpoint behaviour.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

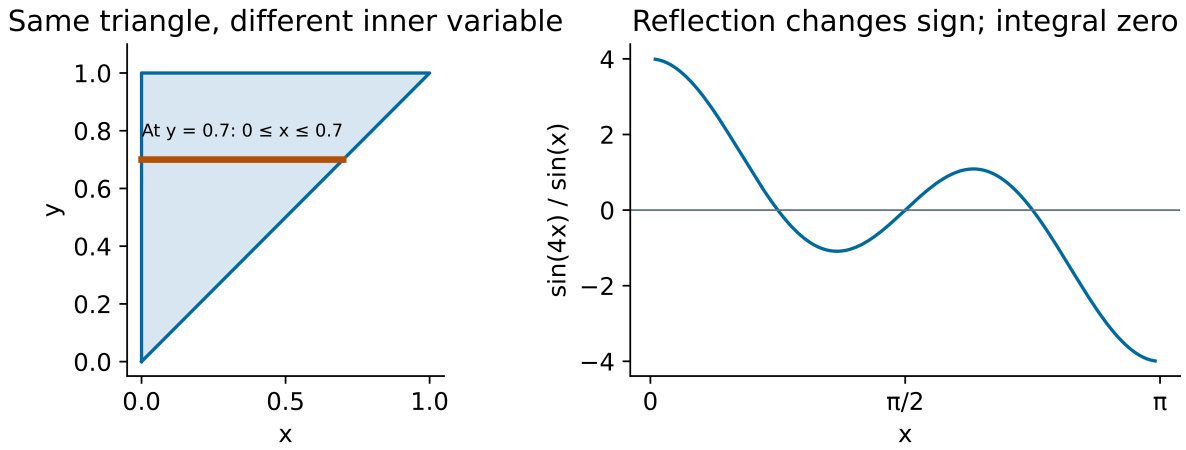
1 Key knowledge and worked examples

integration by parts : An integration identity derived from the product rule with a boundary term.

reflection symmetry : A relation between integrand values at points mirrored across an interval midpoint.

Integration by parts comes from the product rule: $\int u\,dv = uv - \int v\,du$. For definite integrals include the boundary term at both ends. With $I_n(x) = \int_1^x (\ln t)^n\,dt$, choose $u = (\ln t)^n$ and $dv = dt$. For $n \geq 1$ the lower boundary vanishes, giving $I_n = x(\ln x)^n - n I_{n-1}$. A plus sign would violate the differentiated identity. Keep the same lower limit in a recurrence; changing it changes the constants.

For $I_2(x) = \int_1^x (\ln t)^2 dt$, the recurrence gives $x(\ln x)^2 - 2I_1(x)$, with $I_1(x) = x \ln x - x + 1$. Therefore $I_2 = x[(\ln x)^2 - 2\ln x + 2] - 2$. The constant -2 ensures $I_2(1) = 0$. Separately, reversing the triangular $e^{(y^2)}$ integral produces a factor y from the inner x-length; dropping that factor would restore the original difficulty and change the value.



Swap the order of integration by reading the limits from the same region; symmetry can make an integral zero

2 Practice

2.1 If I_n(x)=∫_1^x (ln t)^n dt for n≥1, which recurrence is correct? [1]

- **A** x(ln x)^n+nI_(n−1)
- **B** x(ln x)^n−nI_(n−1)
- **C** (ln x)^n−I_(n−1)
- **D** xI_(n−1)
- **E** nI_(n−1)

2.2 After reversing ∫_0^1 ∫_x^1 e^(y^2) dy dx, what integral remains? [1]

- **A** ∫_0^1 e^(y^2) dy
- **B** ∫_0^1 y e^(y^2) dy
- **C** ∫_0^1 (1−y) e^(y^2) dy
- **D** ∫_0^1 y^2 e^(y^2) dy
- **E** ∫_0^1 e^y dy

2.3 Evaluate $\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$ by reversing order. Describe the region before and after reversal. Derive a formula for $I_2(x) = \int_1^x (\ln t)^2\,dt$, $x > 0$. [4]

3 Exam-style questions

3.1 For positive integer m, evaluate $\int_0^{\pi} \sin(2mx)/\sin x\,dx$. Justify endpoint integrability before invoking symmetry. [4]

Solutions

2.1 B. Integration by parts gives uv minus the remaining integral. The lower boundary is zero for n≥1. The plus-sign choice violates integration by parts; the version without x has wrong boundary term. Multiplying by x or by n alone omits the required boundary expression.

2.2 B. At each y, x ranges from 0 to y. The inner integral supplies length y. Reversal gives inner length y. Unit length, one minus y and y squared describe different regions; e to y changes the given integrand.

2.3 The region is $0 \le x \le y \le 1$. In the reversed order, y ranges from zero to one and x from zero to y. Continuity on the compact triangle permits order reversal, giving $\int_0^1 y e^{y^2}\,dy = (e-1)/2$. Integration by parts with $u = (\ln t)^2$, $dv = dt$ gives $I_2 = x(\ln x)^2 - 2I_1$. Since $I_1 = x\ln x - x + 1$,

$$I_2 = x[(\ln x)^2 - 2\ln x + 2] - 2.$$

Differentiation checks the integrand; substituting x=1 checks the constant.

3.1 The endpoint limits are 2m at zero and -2m at pi, by the first-order sine limit. Both singularities are removable, so the continuously extended integrand is integrable. Reflection $x \mapsto \pi - x$ preserves the denominator and reverses the numerator. The integral equals its negative and therefore is zero. Symmetric cancellation would not establish convergence for a genuinely divergent endpoint; the removable limits are essential.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.