

27.1 Implicit differentiation and the inverse Jacobian

Name: _____ Class: _____ Date: _____ Total: 10 marks

KEY WORDS Jacobian matrix • local inverse

Objective

Differentiate a coupled implicit system as a linear system; Use a nonzero Jacobian determinant to justify a local inverse; Distinguish inverse partial derivatives from scalar reciprocal rules.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

Jacobian matrix : The matrix of first partial derivatives of a vector-valued map.

local inverse : An inverse defined on neighbourhoods of a particular input and output point.

For a differentiable map $F(u,v)=(x,y)=(f(u,v),g(u,v))$, its Jacobian is $J=[[f_u,f_v],[g_u,g_v]]$. Small changes satisfy $[dx,dy] \text{ (transpose)}=J[du,dv] \text{ (transpose)}$ to first order. To find u_x while holding y fixed, differentiate both defining equations with respect to x : $f_u u_x+f_v v_x=1$ and $g_u u_x+g_v v_x=0$. These are a coupled linear system, not two independent scalar inverse rules.

Take $x=u^2+v$ and $y=u-v$. At $(u,v)=(1,0)$, the output is $(1,1)$, and $J=[[2,1],[1,-1]]$ has determinant -3 . The inverse is $[[1/3,1/3],[1/3,-2/3]]$, so $u_x=1/3$ and $v_y=-2/3$ there. Multiplying J by this inverse gives the identity. The reciprocal $1/f_u=1/2$ is not u_x because changing u also requires changing v to keep y fixed.

Coupled sensitivity: inverse entries differ from reciprocals

Forward map	Jacobian	Inverse Jacobian
$x = u + v; y = u + 2v$	$[[1, 1], [1, 2]]$	$[[2, -1], [-1, 1]]$
$x = u^2 + v; y = u - v$ at $(u, v) = (1, 0)$	$[[2, 1], [1, -1]]$	$[[1/3, 1/3], [1/3, -2/3]]$

Invert the whole Jacobian matrix; its entries are not the reciprocals of the forward entries

2 Practice

2.1 For $x=u+v,y=u+2v$, what is u_x when y is held fixed? [1]

- A 0
- B $1/2$
- C 1
- D 2
- E -1

2.2 $J=[[2,1],[1,-1]]$. What is v_y in the inverse derivative matrix? [1]

- A $-2/3$
- B $-1/2$
- C $1/3$
- D $1/2$
- E 2

2.3 Let $x = u^2 + v, y = u - v$. At $(u, v) = (1, 0)$, find the inverse Jacobian and the four inverse partial derivatives. Check by multiplication. [4]

3 Exam-style questions

3.1 For $F(u, v) = (u^2 - v^2, 2uv)$, decide whether its derivative is invertible away from the origin and whether F is globally injective on the punctured plane. What does a zero determinant at the origin establish by itself? [4]

Solutions

2.1 D. The inverse matrix of $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. Its top-left entry is $u_x=2$. Holding y fixed forces u and v to change together. The top-left inverse entry is two, ruling out zero, one-half, one and minus one; one-half would invert a single partial while fixing the wrong variable.

2.2 A. $\det J = -3$. The bottom-right inverse entry is $2/(-3) = -2/3$. The determinant is minus three and the required cofactor is two. The other choices use a different entry, miss the determinant or reverse its sign.

2.3 The corresponding output is $(1, 1)$. The Jacobian is $J = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ with determinant -3 , nonzero. Continuous partials give a differentiable local inverse. Thus

$$J^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}.$$

Read rows as (u_x, u_y) and (v_x, v_y) , all at output $(1, 1)$. Multiplication gives the identity. In particular $u_x = 1/3$, not $1/x_u = 1/2$: v changes to keep y fixed.

3.1 $\det DF = 4(u^2 + v^2) > 0$ away from zero, so the inverse-function theorem supplies a local inverse at each such point. But $F(u, v) = F(-u, -v)$, so it is not globally injective on the punctured plane. A vanishing determinant at zero means this theorem cannot guarantee a differentiable local inverse there. It is not, by itself, a general proof that no local set-theoretic inverse can exist for any map; for example the scalar cube map is bijective despite derivative zero at zero.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.