GRE SUBJECT TEST · GRE MATHEMATICS • EXERCISE SHEET

26.1 Weak compositions, loop invariants and flowchart tracing

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS weak composition • loop invariant

Objective

Count indistinguishable allocations with nonnegative or positive constraints; Trace a flowchart in statement order with correct reset behaviour; Use a loop invariant and a progress measure to justify an algorithm.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

weak composition : An ordered allocation of a total into nonnegative integer parts.

loop invariant : A property preserved before and after each iteration of a loop.

The number of nonnegative integer solutions of $x_1+\cdots+x_k=n$ is $C(n+k-1,k-1)$, for $n\geq 0$ and $k\geq 1$. Represent n identical stars separated into k named boxes by k−1 bars; adjacent bars and end bars allow zero entries. Boxes are distinguishable, objects are not. Counting distinct tokens would instead assign each token independently, giving k^n before other constraints. Decide the model from what the objects and recipients represent, not from a familiar-looking binomial coefficient.

Distribute 11 identical tokens among 4 named boxes. Nonnegative allocations number $C(14,3)=364$; all-positive allocations number $C(10,3)=120$. Therefore 244 allocations have at least one empty box. For the odd-sum test with N=10, S progresses 0,1,4,9,16; it overshoots and rejects 10. For N=9 it exits at S=9 after three additions. The next addends are 1,3,5,7,9; sum and next-addend columns must not be confused.

Odd-sum loop: S = m², B = 2m + 1 before the next addition

Iterations m	Sum S	Next addend B
0	0	1
1	1	3
2	4	5
3	9	7
4	16	9

The loop invariant $S = m^2$ holds before each addition

2 Practice

2.1 How many ways put 11 identical tokens into 4 named boxes, allowing empty boxes? [1]

- **A** 44
- **B** 120
- **C** 244
- **D** 364
- **E** 4^{11}

2.2 Starting from S=0, add successive odd numbers 1,3,5,… until S≥10. What is S at exit? [1]

- **A** 9
- **B** 10
- **C** 11
- **D** 16
- **E** 25

2.3 Count nonnegative solutions of $a + b + c = 8$ with $a \geq 2$. How many also have $b, c \geq 1$? How many of the first set have at least one of b,c equal to zero? [4]

3 Exam-style questions

3.1 Trace this algorithm for N=15 and N=16: initialise S=0, k=1; while S<N, execute S=S+k, then k=k+2; report whether S=N. State an invariant, and justify termination for integer $N \geq 0$. [4]

Solutions

2.1 D. There are 11 stars and 3 bars. Choose the 3 bar positions among 14 positions: C(14,3)=364. Stars and bars gives C(14,3). Forty-four, 120 and 244 do not count that arrangement; 4 to the eleventh counts distinguishable tokens and therefore a different sample space.

2.2 D. The successive sums are 1,4,9,16. The first sum at least 10 is 16, so the test overshoots rather than declaring 10 square. Nine is the previous sum; ten and eleven are not square partial sums; twenty-five is reached only if the loop continues after its condition becomes false.

2.3 Put $u = a - 2$. Then $u + b + c = 6$ has $\binom{8}{2} = 28$ nonnegative solutions. Requiring b,c positive and subtracting one from each gives total four in three nonnegative variables, counted by $\binom{6}{2} = 15$. The complement count is $28 - 15 = 13$. Directly b=0 gives seven, c=0 gives seven, and both zero is counted twice, giving $7 + 7 - 1 = 13$.

3.1 After m iterations, $S = m^2$ and $k = 2m{+}1$. The trace pairs (S,k) are $(0,1), (1,3), (4,5), (9,7), (16,9)$. Thus 15 is rejected by overshoot and 16 accepted. The invariant holds initially and is preserved since $m^2 + (2m{+}1) = (m{+}1)^2$. The sum grows without bound, so it eventually reaches or exceeds N; for example m=N suffices when N is positive. For N=0 the loop is skipped and equality holds. An invariant alone would not establish termination.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.