

25.1 Integrating factors and nonhomogeneous differential equations

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS integrating factor • particular solution

Objective

Solve first-order linear equations using an integrating factor; Construct the homogeneous and particular parts of a constant-coefficient solution; Handle resonance and verify the result in the original equation.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

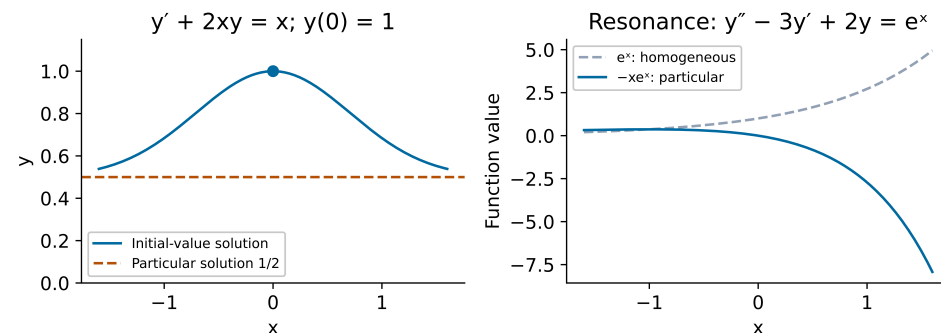
1 Key knowledge and worked examples

integrating factor : A multiplier that turns a first-order linear equation into a product derivative.

particular solution : One solution supplying the specified nonhomogeneous forcing.

A first-order linear equation has the form $y' + p(x)y = q(x)$ on an interval where its coefficients are continuous. It need not be separable. Set $\mu(x) = \exp(\int p(x) dx)$. Since $\mu' = p\mu$, multiplying gives $(\mu y)' = \mu q$, hence $y = \mu^{-1}(\int \mu q dx + C)$. Choose any convenient antiderivative for p ; its integration constant only rescales μ and cancels from the solution. Apply initial data after integration, and keep any singular coefficient points outside the chosen interval.

For $y'' - 3y' + 2y = 4$, the characteristic roots are 1 and 2. A constant particular solution $y_p = 2$ gives $2y_p = 4$, so $y = C_1 e^x + C_2 e^{2x} + 2$. For $y' + 2xy = x$ and $y(0) = 1$, the separate first-order solution is $y = (1 + e^{-x^2})/2$. These illustrate distinct methods; neither equation becomes homogeneous just because its left side is linear.



When the forcing term solves the homogeneous equation, multiply the trial solution by x

2 Practice

2.1 Which integrating factor works for $y' + 3y = x$? [1]

- A x^3
- B e^{3x}
- C e^{-3x}
- D $3x$
- E $1/x$

2.2 For $y'' - 3y' + 2y = 4$, which constant is a particular solution? [1]

- A -4
- B -2
- C 0
- D 2
- E 4

2.3 Solve $y' + 2y = e^x$, $y(0) = 1$, by an integrating factor. Check the answer directly. [4]

3 Exam-style questions

3.1 Solve $y'' - 2y' + y = e^x$ with $y(0) = 0, y'(0) = 0$. Explain why neither Ke^x nor Kxe^x is a sufficient particular trial. [4]

Solutions

2.1 B. $p(x)=3$, so $\mu=\exp(\int 3 \, dx)=e^{3x}$. Its derivative is 3μ . A valid integrating factor has logarithmic derivative three. x^3 , $3x$ and $1/x$ do not; e^{-3x} has the opposite logarithmic derivative.

2.2 D. Both derivatives vanish for a constant. The equation reduces to $2y=4$, so $y=2$. For constants the derivatives vanish and $2y=4$. Minus four, minus two, zero and four fail this substitution.

2.3 The coefficients are continuous everywhere. The integrating factor is $\mu = e^{2x}$. Multiplication gives $(e^{2x}y)' = e^{3x}$. Integration gives $y = e^x/3 + Ce^{-2x}$. The initial condition yields $C=2/3$, so $y = e^x/3 + 2e^{-2x}/3$. Differentiation gives $y' = e^x/3 - 4e^{-2x}/3$; adding $2y$ gives e^x , and $y(0)=1$.

3.1 The homogeneous root is 1 with multiplicity two, so $y_h = (C_1 + C_2x)e^x$. Both suggested trials already belong to that homogeneous space and are annihilated. Put $y = e^xv$; then $y'' - 2y' + y = e^xv''$. Therefore $v'' = 1$ and $v = x^2/2 + C_2x + C_1$. The two initial conditions give $C_1 = C_2 = 0$, hence $y = x^2e^x/2$. This substitution also checks the forcing without relying only on initial values.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.