

24.1 Vector geometry, projections and oriented area

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS orthogonal • cross product

Objective

Use dot products to classify angles and compute projections; Calculate triangle area and orientation using a cross product; Construct or rule out planar dot-product sign configurations.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

orthogonal : Having a zero dot product in a real inner-product space.

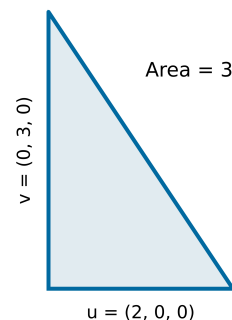
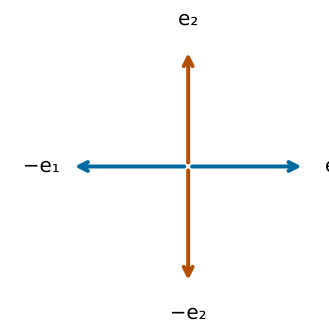
cross product : An oriented perpendicular vector in three dimensions whose magnitude is spanned parallelogram area.

For nonzero real vectors u, v , $u \cdot v = |u||v| \cos \theta$. A positive, zero or negative dot product corresponds to an acute, right or obtuse smaller angle. The scalar projection of v along u is $(v \cdot u)/|u|$; its vector projection is $((v \cdot u)/(u \cdot u))u$. The difference from this projection is orthogonal to u . Do not confuse a projected vector with its signed scalar component. The zero vector is orthogonal to every vector but has no defined angle direction.

With $p=(0,0,0)$, $q=(2,0,0)$, $r=(0,3,0)$, the edges are $u=(2,0,0)$, $v=(0,3,0)$. Their cross product is $(0,0,6)$, giving triangle area 3. For $w=(3,4)$ and $u=(1,0)$, the vector projection is $(3,0)$ and the orthogonal remainder is $(0,4)$. The plane $2x-y+2z=6$ has normal length 3, so its distance from the origin is $6/3=2$.

$$\text{Triangle area} = |u \times v| / 2 = 3$$

Two negative products; four zeros



$$\text{Triangle area} = |u \times v| / 2 = 6 / 2 = 3$$

2 Practice

2.1 Triangle vertices are $(0,0,0), (2,0,0), (0,3,0)$. What is its area?

[1]

- A 1
- B 2
- C 3
- D 6
- E 12

2.2 What is the vector projection of $(3,4)$ onto the line spanned by $(1,0)$?

[1]

- A $(0,4)$
- B $(3,0)$
- C $(1,0)$
- D $(3/5, 4/5)$
- E 5

2.3 Project $v = (2, 3, 6)$ onto $u = (1, 0, 2)$. Find the area of the triangle with vertices O, u and v. Check the projection residual is orthogonal to u. [4]

3 Exam-style questions

3.1 Can four nonzero vectors in $\mathbb{R}^2$ have all six pairwise dot products strictly negative? Give a proof. How does allowing zero products change the possibilities? [4]

Solutions

2.1 C. The edge cross product has magnitude 6. Triangle area is half the spanned parallelogram area. The cross-product norm is six, so six is the parallelogram area and twelve doubles it; one and two fail the same half-base-times-height calculation.

2.2 B. Multiply $(1,0)$ by the coefficient $((3,4) \cdot (1,0))/((1,0) \cdot (1,0))=3$. $(0,4)$ is the orthogonal residual; $(1,0)$ is a direction rather than the projected length; the unit version of v projects onto no stated line; five is a scalar norm, not a vector projection.

2.3 $v \cdot u = 14$ and $u \cdot u = 5$, so the vector projection is $(14/5)u = (14/5, 0, 28/5)$. The residual is $(-4/5, 3, 2/5)$, whose dot product with u is zero. The cross product is $u \times v = (-6, -2, 3)$ with norm seven. Triangle area is half the parallelogram area, or $7/2$. Both edge vectors start at O ; crossing unrelated position vectors would not generally compute the requested triangle.

3.1 Order their directions around the circle. Every consecutive gap must exceed $\pi/2$: if a gap were at most $\pi/2$, that pair would have nonnegative dot product. Four such gaps would sum to more than 2π , impossible. Therefore not all six products can be strictly negative. With nonpositive products, $e_1, e_2, -e_1, -e_2$ works: consecutive pairs are perpendicular and opposite pairs have negative products. A zero vector cannot satisfy strict negativity.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.