20.1 Set images, equivalence relations and logical negation

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS preimage • equivalence relation

Objective

Prove image identities and distinguish inclusion from equality; Check reflexivity, symmetry and transitivity with explicit cases; Negate implications and quantified statements without changing their scope.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

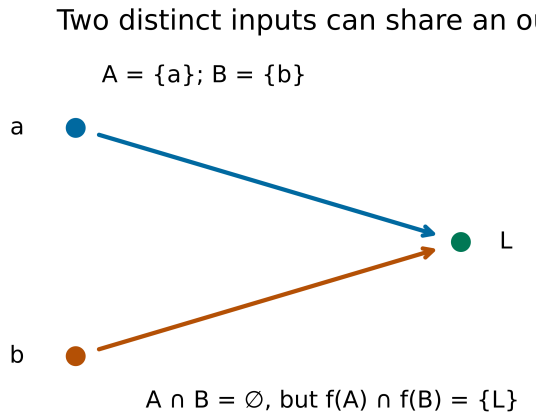
1 Key knowledge and worked examples

preimage : The set of inputs whose outputs lie in a specified target set.

equivalence relation : A relation that is reflexive, symmetric and transitive.

For f:X→Y and A⊆X, the image f(A) consists of all outputs f(a) with a in A. A point is in f(A∪B) exactly when it comes from A or B, so f(A∪B)=f(A)∪f(B). If A⊆B, then f(A)⊆f(B). For an intersection only f(A∩B)⊆f(A)∩f(B) is automatic: the same output may come from different inputs. Use $f(x)=x^2$, A={−1}, B={1}; the left image is empty while the right intersection is {1}.

Let f map both a and b to label L. With A={a}, B={b}, f(A∩B)=∅ but f(A)∩f(B)={L}. For xRy defined by x=y or x=−y on R, reflexivity and symmetry follow directly, and two sign changes still give z=±x, proving transitivity. The classes are {x,−x}, with {0} a singleton. A false statement “every stored file has a hash” means at least one stored file has no hash; it does not mean all files lack hashes.



f(A ∩ B) can be empty while f(A) ∩ f(B) is not: images do not preserve intersections

2 Practice

2.1 Which identity holds for every function f and subsets A,B of its domain? [1]

- **A** f(A∩B)=f(A)∩f(B)
- **B** f(A∪B)=f(A)∪f(B)
- **C** f(A)=A
- **D** f(A)∩f(B)=∅ whenever A∩B=∅
- **E** f(A)⊆A

2.2 What is the negation of P⇒(Q∧R)? [1]

- **A** ¬P∧Q∧R
- **B** ¬P⇒(¬Q∧¬R)
- **C** P∧(¬Q∨¬R)
- **D** P∨¬Q∨¬R
- **E** P∧¬Q∧¬R

2.3 Prove $f(A \cap B) \subseteq f(A) \cap f(B)$. Give a strict-inclusion example. Under what additional assumption on f does equality follow for all A and B? [4]

3 Exam-style questions

3.1 On the integers define aRb when $a - b$ is divisible by 4. Prove it is an equivalence relation and describe every class. Negate: "For every integer n there exists an integer m greater than n with property P." [4]

Solutions

2.1 B. An output of an input in the union comes from at least one subset. Intersection equality can fail when distinct inputs share an output. A and D fail for a constant map on distinct inputs. C and E compare domain subsets with outputs without any identity-map or containment assumption.

2.2 C. Failure of the implication requires a true premise and a false conjunction. De Morgan changes the latter into a disjunction. A makes the premise false; B is not the negation of the original implication; D allows a false premise. E requires both conclusions to fail, though failure of either is enough.

2.3 If $y = f(x)$ for $x \in A \cap B$, then x belongs to each set, so y belongs to each image. For strictness, map distinct a,b to the same c and take $A = \{a\}$, $B = \{b\}$. The left image is empty and the right intersection is $\{c\}$. If f is injective and $y = f(a) = f(b)$ with $a \in A, b \in B$, injectivity gives a=b in the intersection, proving reverse inclusion. Conversely the singleton example shows that equality for all pairs forces injectivity.

3.1 Reflexivity uses $a - a = 0$; symmetry uses the negative of a multiple of four; transitivity uses the sum of two such multiples. Classes are the four residues modulo four; the class of a is $a + 4\mathbb{Z}$. The negation is: there exists an integer n such that every integer m greater than n fails P. Equivalently $\exists n \, \forall m \, (m > n \Rightarrow \neg P(m))$. It does not assert that P fails for every integer, nor that m is at most n for every m.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.