

20.1 Set images, equivalence relations and logical negation

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS preimage • equivalence relation

Objective

Prove image identities and distinguish inclusion from equality; Check reflexivity, symmetry and transitivity with explicit cases; Negate implications and quantified statements without changing their scope.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

preimage : The set of inputs whose outputs lie in a specified target set.

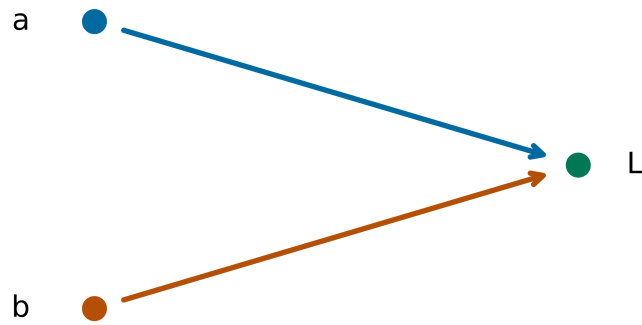
equivalence relation : A relation that is reflexive, symmetric and transitive.

For $f: X \rightarrow Y$ and $A \subseteq X$, the image $f(A)$ consists of all outputs $f(a)$ with a in A . A point is in $f(A \cup B)$ exactly when it comes from A or B , so $f(A \cup B) = f(A) \cup f(B)$. If $A \subseteq B$, then $f(A) \subseteq f(B)$. For an intersection only $f(A \cap B) \subseteq f(A) \cap f(B)$ is automatic: the same output may come from different inputs. Use $f(x) = x^2$, $A = \{-1\}$, $B = \{1\}$; the left image is empty while the right intersection is $\{1\}$.

Let f map both a and b to label L . With $A = \{a\}$, $B = \{b\}$, $f(A \cap B) = \emptyset$ but $f(A) \cap f(B) = \{L\}$. For xRy defined by $x=y$ or $x=-y$ on $\mathbb{R}$, reflexivity and symmetry follow directly, and two sign changes still give $z = \pm x$, proving transitivity. The classes are $\{x, -x\}$, with $\{0\}$ a singleton. A false statement “every stored file has a hash” means at least one stored file has no hash; it does not mean all files lack hashes.

Two distinct inputs can share an output

$$A = \{a\}; B = \{b\}$$



$$A \cap B = \emptyset, \text{ but } f(A) \cap f(B) = \{L\}$$

$f(A \cap B)$ can be empty while $f(A) \cap f(B)$ is not: images do not preserve intersections

2 Practice

2.1 Which identity holds for every function f and subsets A, B of its domain? [1]

- A $f(A \cap B) = f(A) \cap f(B)$
- B $f(A \cup B) = f(A) \cup f(B)$
- C $f(A) = A$
- D $f(A) \cap f(B) = \emptyset$ whenever $A \cap B = \emptyset$
- E $f(A) \subseteq A$

2.2 What is the negation of $P \Rightarrow (Q \wedge R)$? [1]

- A $\neg P \wedge Q \wedge R$
- B $\neg P \Rightarrow (\neg Q \wedge \neg R)$
- C $P \wedge (\neg Q \vee \neg R)$
- D $P \vee \neg Q \vee \neg R$
- E $P \wedge \neg Q \wedge \neg R$

2.3 Prove $f(A \cap B) \subseteq f(A) \cap f(B)$. Give a strict-inclusion example. Under what additional assumption on f does equality follow for all A and B ? [4]

3 Exam-style questions

3.1 On the integers define aRb when $a - b$ is divisible by 4. Prove it is an equivalence relation and describe every class. Negate: “For every integer n there exists an integer m greater than n with property P .” [4]

Solutions

2.1 B. An output of an input in the union comes from at least one subset. Intersection equality can fail when distinct inputs share an output. A and D fail for a constant map on distinct inputs. C and E compare domain subsets with outputs without any identity-map or containment assumption.

2.2 C. Failure of the implication requires a true premise and a false conjunction. De Morgan changes the latter into a disjunction. A makes the premise false; B is not the negation of the original implication; D allows a false premise. E requires both conclusions to fail, though failure of either is enough.

2.3 If $y = f(x)$ for $x \in A \cap B$, then x belongs to each set, so y belongs to each image. For strictness, map distinct a, b to the same c and take $A = \{a\}$, $B = \{b\}$. The left image is empty and the right intersection is $\{c\}$. If f is injective and $y = f(a) = f(b)$ with $a \in A, b \in B$, injectivity gives $a=b$ in the intersection, proving reverse inclusion. Conversely the singleton example shows that equality for all pairs forces injectivity.

3.1 Reflexivity uses $a - a = 0$; symmetry uses the negative of a multiple of four; transitivity uses the sum of two such multiples. Classes are the four residues modulo four; the class of a is $a + 4\mathbb{Z}$. The negation is: there exists an integer n such that every integer m greater than n fails P . Equivalently $\exists n \forall m (m > n \Rightarrow \neg P(m))$. It does not assert that P fails for every integer, nor that m is at most n for every m .

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.