

2.1 Multivariable calculus and vector analysis

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS gradient • Jacobian

Objective

Compute partial derivatives, gradients and directional derivatives; Use multiple integrals and coordinate changes; Use derivative conditions to distinguish planes from curved surfaces.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

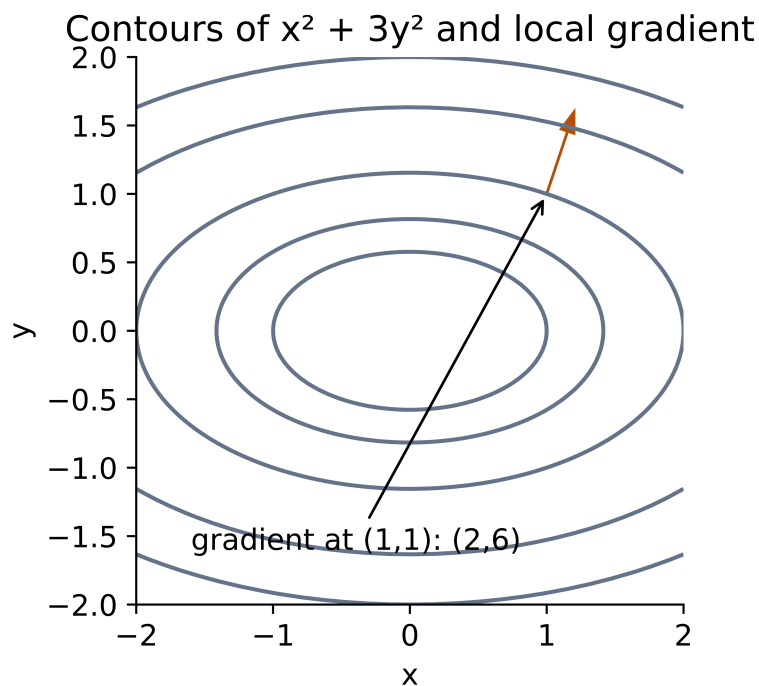
1 Key knowledge and worked examples

gradient : Vector of partial derivatives.

Jacobian : The local area or volume scaling in a coordinate change.

A partial derivative changes one coordinate while fixing the others. For $f=x^2+3xy$, $f_x=2x+3y$ and $f_y=3x$. The gradient collects these derivatives; if f is differentiable, the directional derivative along a unit vector v is $\text{grad } f \cdot v$. Normalise the direction before taking this dot product. A direction vector of length two would double the answer if used without normalisation.

For $f(x,y)=x^2+3y^2$, $\text{grad } f=(2x,6y)$. At $(1,1)$ this is $(2,6)$. In direction $(3,4)$, the unit vector is $(3/5,4/5)$, giving directional derivative $2 \cdot 3/5 + 6 \cdot 4/5 = 6$.



The gradient $(2, 6)$ at $(1, 1)$ is perpendicular to the contour and points uphill

2 Practice

2.1 Partial derivative with respect to x of x^2y is:

[1]

- **A** x^2
- **B** $2xy$
- **C** $2x$

2.2 The polar area element is:

[1]

- **A** $dr \, d\theta$
- **B** $r \, dr \, d\theta$
- **C** $r^2 \, dr \, d\theta$

2.3 For $f(x, y) = x^2 + xy + 2y^2$, find the directional derivative at $(1, -1)$ towards $(3, 4)$. Evaluate $\iint_D (x^2 + y^2) dA$ for the unit disk D . [4]

3 Exam-style questions

3.1 Define $g(x, y) = xy/\sqrt{x^2 + y^2}$ away from the origin and $g(0, 0) = 0$. Both partial derivatives at the origin are zero. Is g differentiable there? Is it continuous? [4]

Solutions

2.1 B. Treat y as constant. A differentiates the wrong factor; C drops the fixed factor y . At $y=0$ every derivative value is zero, consistently with $2xy$.

2.2 B. The Jacobian of the polar transformation is r . A omits the local scale r ; C is the radial power for a three-dimensional spherical volume, with an additional angular factor, not a planar area element.

2.3 The polynomial is differentiable. Normalise the direction: $u = (3/5, 4/5)$. The gradient is $(2x + y, x + 4y)$, hence $(1, -3)$ at the point.

$$D_u f(1, -1) = \nabla f(1, -1) \cdot u = 1 \cdot 3/5 - 3 \cdot 4/5 = -9/5.$$

Polar coordinates give $x^2 + y^2 = r^2$ and $dA = r \, dr \, d\theta$:

$$I = \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta = \pi/2.$$

3.1 Along each axis $g = 0$, so both partials are zero. Continuity follows from $|xy| \leq (x^2 + y^2)/2$, which gives $|g(x, y)| \leq \sqrt{x^2 + y^2}/2 \rightarrow 0$. If differentiable, its linear derivative would be zero. Along $x = y = t \neq 0$,

$$\frac{|g(t, t)|}{\sqrt{t^2 + t^2}} = \frac{1}{2}.$$

The required remainder ratio does not tend to zero. Thus continuity and existing partial derivatives do not establish differentiability.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.