

18.1 Coordinate changes and vector integral theorems

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS divergence • conservative field

Objective

Transform double and triple integrals with their Jacobians; Apply Green, Stokes and divergence theorems with correct orientation; Check domain singularities before asserting path independence.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

divergence : The sum of a vector field's coordinate-wise partial derivatives measuring local outward flow.

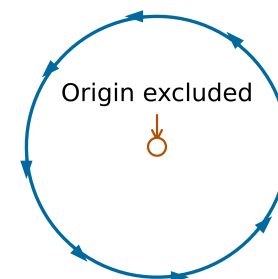
conservative field : A vector field equal to a scalar potential gradient with path-independent line integrals.

Changing variables scales area or volume by the absolute Jacobian determinant. Polar coordinates use $r dr d\theta$; cylindrical coordinates use $r dr d\theta dz$; spherical coordinates with ϕ measured from the positive z -axis use $\rho^2 \sin(\phi) d\rho d\phi d\theta$. State angle conventions and transform both the integrand and the region. A missing Jacobian changes a uniform density integral into the wrong physical quantity.

For $F=(x,y,z)$, divergence is 3. The outward flux through the sphere of radius 2 is therefore 3 times its volume, or $3 \cdot (4\pi \cdot 2^3/3)=32\pi$. This avoids a surface parameterisation. For the planar field $(-y,x)$, Green's theorem gives counterclockwise unit-circle circulation as integral of $1-(-1)=2$ over the disk, hence 2π . Reversing the orientation changes the circulation sign.

A hole can prevent a global potential

$$F = (-y/r^2, x/r^2)$$



Unit-circle circulation = 2π , despite zero curl

Zero curl but circulation 2π : the hole at the origin prevents a global potential

2 Practice

2.1 What is the polar-coordinate area element? [1]

- A $dr d\theta$
- B $r dr d\theta$
- C $r^2 dr d\theta$
- D $\sin(\theta) dr d\theta$
- E $r^{-1} dr d\theta$

2.2 For $F=(x,y,z)$, what is the outward flux through the unit sphere? [1]

- A 0
- B π
- C 2π
- D 4π
- E 12π

2.3 Evaluate $\iint_D (x^2 + y^2) dA$ on $1 \leq x^2 + y^2 \leq 4$. Find the outward flux of $F = (x, y, z)$ across the unit sphere. [4]

3 Exam-style questions

3.1 Let $G = (-y/(x^2 + y^2), x/(x^2 + y^2))$ on the punctured plane. Its scalar curl is zero there. Evaluate its circulation on the counterclockwise unit circle. Explain why Green's theorem does not make it zero. [4]

Solutions

2.1 B. The absolute Jacobian of $(r \cos(\theta), r \sin(\theta))$ is r . Unit scale, r squared, $\sin \theta$ and inverse r do not equal the polar area Jacobian r . The sine factor belongs to spherical coordinates with different variables.

2.2 D. Divergence is 3 and unit-ball volume is $4\pi/3$, so outward flux is 4π . Divergence three times ball volume 4π over three is 4π . Twelve π uses sphere area in place of volume; zero and the smaller π multiples omit part of the flux.

2.3 The annulus has polar bounds $1 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. Including the Jacobian,

$$I = \int_0^{2\pi} \int_1^2 r^3 dr d\theta = 15\pi/2.$$

F is continuously differentiable on the whole ball and has divergence three. The divergence theorem gives outward flux $3(4\pi/3) = 4\pi$. Inward orientation would reverse its sign.

3.1 Parameterise $r(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$. Then $G(r(t)) = (-\sin t, \cos t) = r'(t)$, so $G \cdot r' = 1$ and the integral is 2π . The field is undefined at the origin inside the disk. Green's theorem requires the needed smoothness on a neighbourhood of the region, not just on its boundary. Zero curl on a punctured domain does not establish a global potential; a closed nonexact field can circulate around its hole.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.